St John's College

Physics Example Classes

CLASS 1 – MECHANICS

This document contains the material I'll be covering in my first examples class, covering your michaelmas "mechanics" course.

It consists of two sections:

- A preliminary blurb on how to generally tackle mechanics problems, especially problems that make you wonder whether you've picked up the IB examples sheet by mistake!
- A selection of questions, some from past tripos papers. I spent many hours making sure they cover every aspect of the course (Also, I never pick questions from tripos papers later than 2004, so that you have some left to practice by yourselves. If there was a topic that was not examined before 2004, made up a new tripos-style question).

You will get infinitely more out of the examples class if you attempt those problems before the class. That said, I'm painfully aware of what it's like to be a IA Natsci, and you're more than welcome to come even if you haven't looked at the questions before. In case you do choose to look at them, I've given them arbitrary difficulty ratings (*, ** or ***) to give you an idea… [All, however, are at least of tripos standard].

I will hand out a full set of typed solutions **to those who attend the session**.

Questions, comments, feedback, suggestions, personals stories or jokes are always gratefully accepted, at guetta@cantab.net.

Daniel

Section 1 – How to do Mechanics Problems

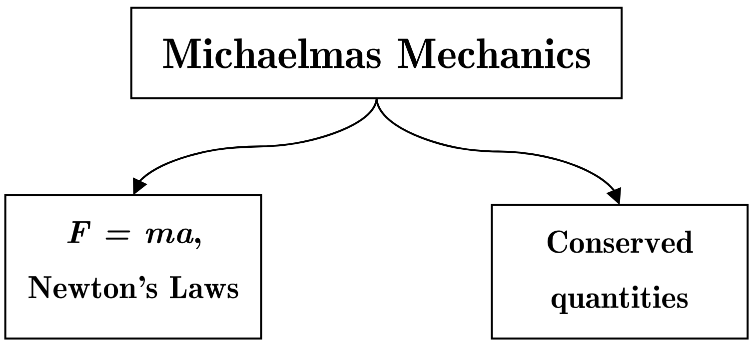
Usually, difficult mechanics problems are difficult for one of two reasons:

- It's difficult to identify what concept/tool you need to use ($F = ma$, conservation laws, etc…)
- Once you've identified the tool, it's pretty difficult to apply it to the problem at hand.

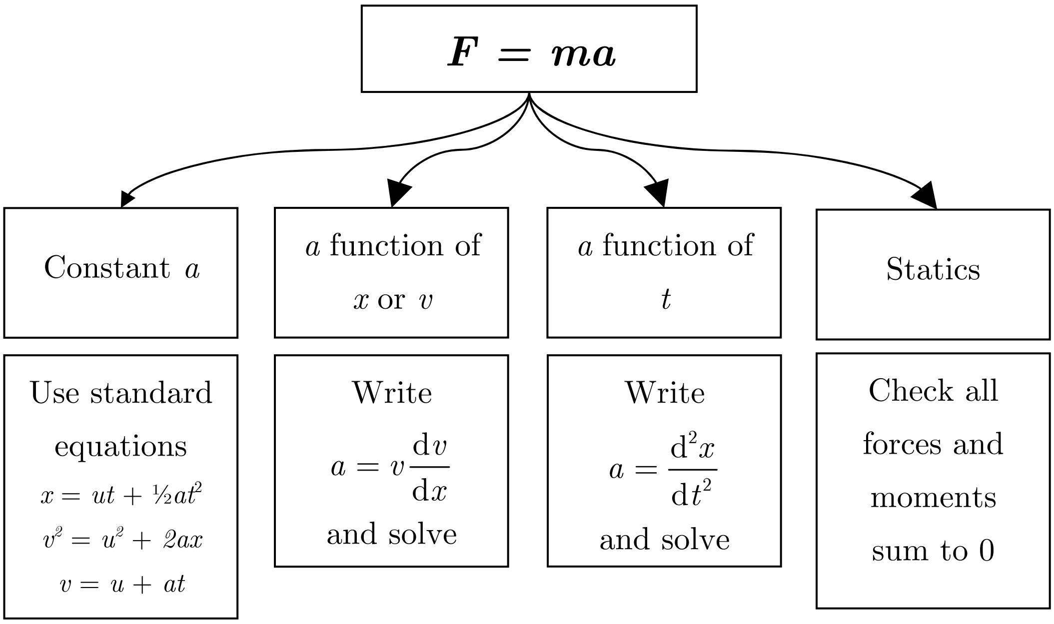
Both will come with practice, and hopefully this section can act as a reference to help you along the way.

The first thing you'll need to do when you approach a problem is check whether you need to split it into lots of smaller problems. For example, you might be set a question on a ball that is given an impulse, set rolling and then brought to rest by friction. There are actually three parts to this problem: (1) ball is set moving (2) ball rolls (3) ball brought to rest. You will need to deal with each of these parts using a different method.

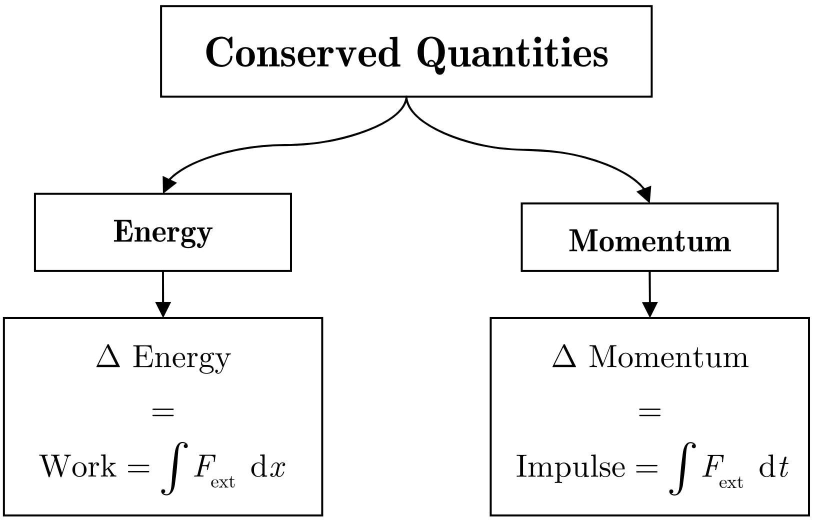
Once you've done that, look at each part separately, and decide how you will deal with that part. Basically, everything you've learnt this term can be divided into two different approaches – here they are:



Here's a quick summary of the different situations you might encounter with each approach (*model*):



Next comes conserved quantities – the key thing to remember here is that the name "conserved quantities" is a bit of a joke, because they're often not conserved! We're sometimes interested by processes in which these quantities change – so much so that we give a special *name* to these changes[1]:



Often, the force will be **constant**, and you'll be able to simply take it out of the integral. In that case, work and impulse become

[1] Notice the symmetry here, which you might not have noticed in lectures; both work and impulse are the integral of force, but with respect to different fundamental quantities. This isn't a coincidence, as you'll find out in Part II ☺.

$$\text{Work} = F_{\text{ext}} \int \mathrm{d}x = F_{\text{ext}} \left(x_{\text{initial}} - x_{\text{final}} \right) = F_{\text{ext}} \Delta x$$
$$\text{Impulse} = F_{\text{ext}} \int \mathrm{d}t = F_{\text{ext}} \left(t_{\text{initial}} - t_{\text{final}} \right) = F_{\text{ext}} \Delta t$$

There's one last key-point to make, and it's best illustrated by a mistake people often make (in problems slightly more complex than this one)

Question: A force of magnitude F is applied to a particle of mass 5 kg, which, as a result, accelerates from rest to a speed of 5 m/s in 1 second. Find F.

Answer: Well, OK, so I know that $F = 5a$, because of Newton's Second Law. So I just need to use $F = ma$ to eliminate a and find F! Let's try it

$$F = ma \Rightarrow a = F \, / \, m$$

$$F = 5a \Rightarrow F = 5\frac{F}{m} \Rightarrow F = F$$

Oh dear!

The key mistake in the answer above is that it doesn't take into account the fact there are **two sides** to all physical laws (like $F = ma$). What the answer should have done is

- Find out something about the LHS **fom the question** (ie: that the force is F).
- Find out something about the RHS **from the question** (ie: that $m = 5$ kg and that $a = 5$ m/s^2)

And then ***finally*** use $F = ma$ to set those two bits equal to each other and deduce that

$$F = ma = 5 \times 5 = 25 \text{ N}$$

Laws of physics are **not identities**. You have to find something about each side separately from the question, and *then* set them equal to each other. The answer above only considered the F side and didn't bother thinking of a!

Section 2 – Questions & Solutions

Question 1 – Shorties

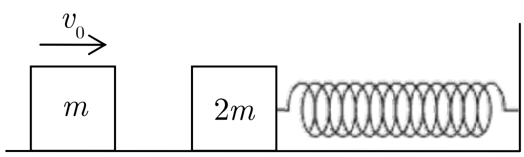
(a) * (*Tripos 1994*) Why does the front end of a car dip upon braking?

(b) ** (*Tripos 1996*) In a poorly maintained train, the thin cavity of a double-glazed window is partially filled with rain water. As the train decelerates along a horizontal track, a passenger notices that the water surface is at an angle of 15° to the horizontal. What is the deceleration of the train?

(c) * (*Tripos 2003*) A massless ladder of length l rests against a smooth wall. The ladder subtended by the ladder and the ground is α. The coefficient of static friction between the ground and the ladder is μ. There is no friction between the ladder and the wall. And undergraduate of mass m starts the climb the ladder. Find an expression for the maximum vertical height above the ground that the undergraduate can reach before the ladder slips.

(d) ** A hot air balloon of mass M is stationary (with respect to the ground) in mid-air. A passenger of mass m climbs out and slides down a rope with constant velocity v with respect to the ground. With what velocity (magnitude and direction) relative to the ground does the balloon move? What happens if the passenger then stops sliding? [*Harder: is energy conserved while the passenger is sliding? Does that make sense? Shouldn't there be some heat lost to friction with the rope?*]

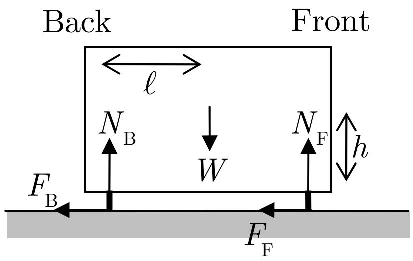
(e) * A stationary block of mass $2m$ lies on a frictionless table and is attached to a spring at its equilibrium position. A block of mass m collides into the stationary block with speed v_0 and sticks to it instantaneously. The blocks continue moving towards the wall, compressing the spring.



What is the magnitude and direction of the total impulse exerted by the spring on the particles from a time just after the collision until the time the spring reaches its shortest length?

Solutions:

(a) At first sight, a pretty difficult question, but it's really not that bad. The key insight here is to realise that this is going to have something to do with **forces**, and so a diagram of the forces on the car as it breaks is what we're going to need. Each wheel is going to experience a normal force and a friction force, and there'll also be gravity acting on the car:



Now, as the car is braking, it's decelerating in the x direction, but it's clearly not rotating. As such, the turning moments about the centre of mass of the car should sum to 0:

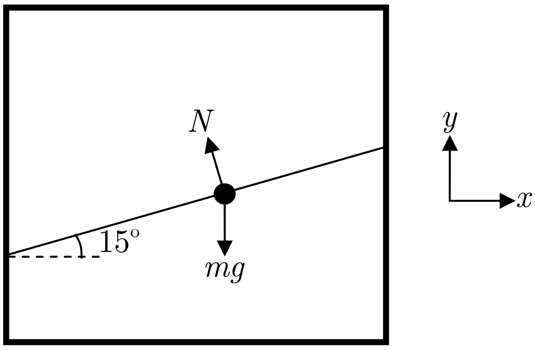
$$F_B h + N_B \ell + F_F h = N_F \ell$$

Diving by ℓ and re-arranging, we have that

$$N_F = N_B + \frac{h}{\ell}\left(F_F + F_B\right)$$

This expression clearly shows us that $N_F > N_B$. However, the normal forces act on a car through spring suspensions, and this therefore means that the spring at the front of the car will have to be *more compressed* than the one at the back, to provide the larger force. Thus, the car dips forwards.

(b) This is a very hard question indeed, because it's hard to see where to begin. Clearly, you're going to need some force diagrams, but the force on *what*? The key insight (which took me a few minute to get) is that you just have to consider a little bit of water of mass m at some point on the surface, and just deal with that:



OK, so let's find the horizontal and vertical forces on this thing

$$\uparrow \qquad N\cos\theta - mg = ma_y$$
$$\rightarrow \qquad -N\sin\theta = ma_x$$

OK, so we've used our force diagram to deal with one side of the $F = ma$ equation. Can we use the problem to deal with the other side? Yup! We know $a_y = 0$, because while the train is decelerating, the slope stays put (it's not wobbling up and down as long as the train is decelerating constantly), and a_x is the amount by which the train is decelerating, which is what we want! So: no

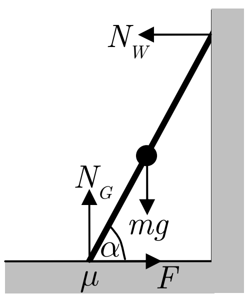
$$N\cos\theta = mg$$
$$-N\sin\theta = ma_x$$

Dividing the bottom equation by the top one:

$$a_x = -g\frac{\sin\theta}{\cos\theta} = -g\tan\theta$$

And we're done! (Which makes sense, by the way – as $\theta \to 90^\circ$, the train is decelerating very fast indeed! And the answer is dimensionally consistent.) They key physical insight is that the slope in the water is needed to provide the (normal) force that decelerates the water.

(c) This is a bog-standard force/statics question. Let's imagine the undergraduate has climbed a distance x from the bottom of the ladder. The forces then look like this:



The situation is static, so we set resultant horizontal forces, vertical forces and turning moments equal to 0:

$$\uparrow \qquad N_G = mg$$
$$\rightarrow \qquad F = N_W$$

Taking turning moments about the point of contact with the ground:

$$mgx\cos\alpha = N_W l\sin\alpha \Rightarrow N_W = \frac{mgx}{l\tan\alpha}$$

Let's grind all of this to obtain an expression for F in terms of α

$$F = \frac{mgx}{l\tan\alpha}$$

Finally, we note that if the ladder is not to slip, then $F \le \mu N_G = \mu mg$. Thus

$$F = \frac{mgx}{l\tan\alpha} \le \mu N_G = \mu mg$$
$$\frac{mgx}{l\tan\alpha} \le \mu mg$$
$$x \le \mu l\tan\alpha$$

This is the *maximum* possible height along the ladder it not to slip. Thus, the maximum vertical height is

$$\boxed{x \le \mu l\tan\alpha\sin\alpha}$$

(d) The first thing you might be tempted to do is consider the *forces* on the balloon and the person as they start sliding down. That would, eventually, lead to the answer, but only after lots of blood and tears. If you're lucky, you'll happen upon the fact that conservation of momentum is actually the way to go here. Originally, the momentum of the system is 0. There is an external force acting on the system (gravity) but it is balanced by the upthrust on the balloon, which must be equal (since the balloon is originally at rest). Thus, there is no external force acting on the system, and the momentum should also be 0 once the person has started sliding down. Thus, if V is the velocity of the balloon

$$mv = \left(M - m\right)V$$
$$V = \frac{mv}{M - m}$$

Thus, the balloon moves upwards at that velocity.

Let's check if energy is conserved – every second, the person drops a distance v and the balloon climbs a distance V. Thus, if energy was conserved we would expect

$$mgv = \left(M-m\right)gV$$
$$mgv = \left(M-m\right)g\frac{mv}{M-m}$$
$$mgv = mgv$$

So energy is indeed conserved.

(e) I included this problem because it demonstrates two key principles of problem solving.

The first principle is that it's crucial to break the problem into its constituent parts – this one involves *two* parts:

1. **The collision**

   The collision is clearly inelastic. During the collision, no external forces are acting, and so momentum is conserved. If we assume the combined blocks are moving at a speed V after the collision, we get:

$$mv_0 = 3mV$$
$$\boxed{V = \frac{v_0}{3}}$$

2. **The compression of the spring**

   To find the impulse, you might be tempted to use

$$I = \int F \text{ d}t$$

   Feed in an explicit expression for the force, and then integrate. You can do that, and it'll give you the answer (you'll know how by the end of this term), but you'll be there for a *long* time.

   And here comes the second principle this question illustrate. You've looked at the RHS of the equation, but have you bothered looking at the LHS? There's actually an incredibly easy way of working out the impulse here – it's just the change in momentum!

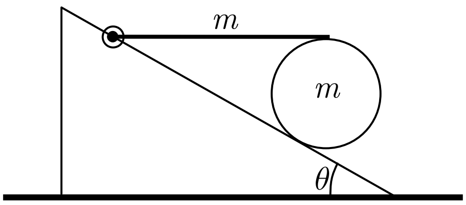
Originally, the block has momentum *3mV*, eventually it has momentum 0! So:

$$\boxed{I = 3mV = mv_0}$$

The direction is clearly to the *left*, to stop the blocks.

**** <u>Question 2</u>**

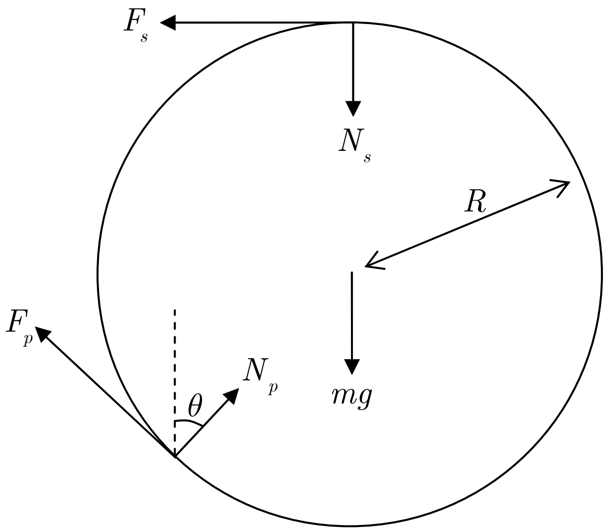
A horizontal stick of mass m has its left end attached to a pivot on a plane inclined at an angle θ, while its right end rests on the top of a cylinder also of mass m which in turns rests on the plane. The coefficient of friction between the cylinder and both the stick and the plane is μ



What is the smallest value of μ (in terms of θ) for which the system doesn't slip anywhere?

Solution:

For starters, it's pretty clear that this is a statics problems, and that you're going to have to sum forces and turning moments. A good place to start is a nice big diagram of the cylinder with all the forces acting on it:



These forces must balance horizontally and vertically:

$$\rightarrow \qquad F_s + F_p \cos\theta = N_p \sin\theta$$
$$\uparrow \qquad N_p \cos\theta + F_p \sin\theta = mg + N_s$$

Turning moments also need to balance about all points. In this case, you can immediately see that taking turning moments around the centre of the cylinder will lead to a particularly nice result:

$$F_p R = F_s R \Rightarrow F_p = F_s$$

(the normal forces drop out because they are parallel to the line between the centre of the cylinder and their point of action). [***Note***: this does **not** imply that $N_s = N_p$, because N only tells us about the *maximum* value friction can reach].

Using this equation, we can eliminate one of the frictions from the equations above (we'll denote $F_p = F_s = F$) and get

$$F\left(1+\cos\theta\right) = N_p \sin\theta$$
$$N_p \cos\theta + F\sin\theta = mg + N_s$$

At this point, we take stock of what we have and we look back at the question. We want to find the **minimum value of** μ. Clearly, this will have something to do with the fact that at each surface $F \leq \mu N$. In other words

$$F \leq \mu N_s$$
$$F \leq \mu N_p$$

So if we could only express F in terms of N_s and N_p only, we might be done. Let's try it:

$\boxed{\text{Expressing } F \text{ in terms of } N_p}$

Here, our work's already done for us – we just use the first equation

$$F = N_p \frac{\sin\theta}{1+\cos\theta}$$

And we know that

$$F = N_p \frac{\sin\theta}{1+\cos\theta} \leq \mu N_p$$
$$\boxed{\frac{\sin\theta}{1+\cos\theta} \leq \mu}$$

So this is our first constraint for μ.

$\boxed{\text{Expressing } F \text{ in terms of } N_s}$

Here, we need to do some fiddling. From the first equation, we have

$$N_p = F\frac{1+\cos\theta}{\sin\theta}$$

Putting that into the second equation, we get

$$F\frac{1+\cos\theta}{\sin\theta}\cos\theta + F\sin\theta = mg + N_s$$

Time for a bit of hardcore re-arranging:

$$F\left(\frac{\cos\theta+\cos^2\theta}{\sin\theta}+\sin\theta\right) = mg + N_s$$

You might not be quite sure what to do here, but it's always a good idea to try and put everything in one fraction – you might get lucky:

$$F\left(\frac{\cos\theta+\cos^2\theta+\sin^2\theta}{\sin\theta}\right) = mg + N_s$$

$$F\left(\frac{1+\cos\theta}{\sin\theta}\right) = mg + N_s$$

$$F = \frac{\sin\theta}{1+\cos\theta}\left(mg + N_s\right)$$

And we know that

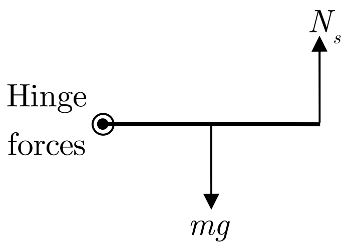
$$\frac{\sin\theta}{1+\cos\theta}\left(mg + N_s\right) \le \mu N_s$$

At this point, however, we're stuck, because the N_s doesn't cancel as nicely as it did in the previous part…

Hum… What do to do? When you reach a point like this in a problem, it's always worth doing two things:

- **Establishing a "wish-list"** – in this case, I'd really, really like to find a way to express N_s in terms of mg – because then I could replace mg by something to do with N_s, and hopefully the above will cancel better.
- **Go back to the problem, and see if there's any extra physical information you've missed out** – in this case, if you're lucky, you'll notice that we did all the equilibrium stuff for the cylinder, but not for the stick!

It seems, therefore, that it's worth doing the equilibrium thing for the stick:



You could start doing forces, etc..., but you can already see the forces at the hinge are going to be a royal pain in the backside. Is there a way we can avoid them? Indeed there is – take turning moments around the hinge (I'll denote the length of the stick by ℓ)!

$$mg\frac{\ell}{2} = N_s\ell \Rightarrow \boxed{mg = 2N_s}$$

We have mg in terms of N_s, which is what was on our wishlist!

So let's go back to our inequality:

$$\frac{\sin\theta}{1+\cos\theta}\left(mg + N_s\right) \le \mu N_s$$
$$\frac{3\sin\theta}{1+\cos\theta}N_s \le \mu N_s$$
$$\boxed{\frac{3\sin\theta}{1+\cos\theta} \le \mu}$$

So we have our second inequality!

Let's summarize the two inequalities we got:

$$\frac{\sin\theta}{1+\cos\theta} \le \mu \qquad \frac{3\sin\theta}{1+\cos\theta} \le \mu$$

Clearly, the second one is more stringent, because the LHS is multiplied by 3. So our answer is

$$\boxed{\boxed{\frac{3\sin\theta}{1+\cos\theta} \le \mu}}$$

*** Question 3**

When you study orbits in IB, you'll come across the (very remarkable) fact that objects orbiting a planet of mass M at a radius r can be modeled as particles moving in a straight line under the effect of the following potential

$$U(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

Where L, m, G and M are constants[2].

Find the equilibrium points of the system above. Are they stable or unstable?

(*Harder and completely optional: in terms of the original orbit problem, what does an equilibrium point mean?*)

Solution:

This is a bog-standard equilibrium problem. Differentiate and set to 0:

$$\frac{\mathrm{d}}{\mathrm{d}r}U = -\frac{L^2}{mr^3} + \frac{GMm}{r^2} = 0$$

$$GMmr = \frac{L^2}{m}$$

$$\boxed{r = \frac{L^2}{GMm^2}}$$

To classify the equilibrium, we could differentiate again and feed in, but we could also be clever and note that

- As $r \to 0$, $U \to +\infty$, because $1/r^2$ is bigger than $1/r$ when r is close to 0.
- As $r \to \infty$, $U \to 0$
- There's only one stationary point between those two

It is therefore clear that the stationary point is a minimum – if you visualize the function, it couldn't be otherwise. (Well, it could be a point of inflexion…)

[In terms of the original orbit, points of stable r imply circular orbits, because r stays constant all the time. When you study orbits next term, try feed in $L = mrv$ into $r = L^2 / GMm^2$ and see why it makes sense!]

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[2] For future reference: L is the angular momentum of the orbiting body, which you'll learn about next term, m is the mass of the orbiting body and M is the mass of the planet.

<u> Question 4</u>**

A block of mass m is placed on a smooth plane inclined at an angle θ to the horizontal, and is pushed down along the plane at a very high speed v_0. Under those conditions, the block is subject to air resistance of magnitude $F = bv^2$ parallel to its direction of motion but in the opposite direction (ie: resisting the motion).

Find the terminal velocity of the block, and the velocity of the block after it has traveled a distance x from its starting point.

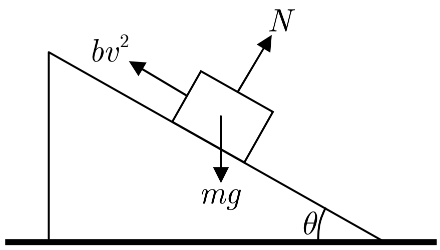
You might find the following integral useful:

$$\int \frac{x}{A - Bx^2}\ \mathrm{d}x = -\frac{1}{2B}\ln\left|A - Bx^2\right|$$

If you want to try to prove it, just substitute $u = A - Bx^2$.

Solution:

This is an $F = ma$ problem. Let's check out the forces acting on the block while it's falling. Since the block is moving directly down the plane, the friction will be acting directly opposite that:



Using $F = ma$ parallel to the plane, and taking downwards as positive, we get

$$mg\sin\theta - bv^2 = ma$$

$$a = g\sin\theta - \frac{b}{m}v^2$$

The terminal velocity is reached when the block is no longer accelerating; when $a = 0$. Thus:

$$g\sin\theta = \frac{b}{m}v_{\text{terminal}}^2 \Rightarrow \boxed{v_{\text{terminal}} = \sqrt{\frac{mg\sin\theta}{b}}}$$

To find v in terms of x, we follow the tips in the introductory section of this handout, and we write $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$. We then get:

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = g\sin\theta - \frac{b}{m}v^2$$

We need to separate variables:

$$\frac{v}{g\sin\theta - \frac{b}{m}v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

Finally, we can integrate with respect to x

$$\int \frac{v}{g\sin\theta - \frac{b}{m}v^2}\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = \int 1\ \mathrm{d}x$$

$$\int_{v_0}^{V} \frac{v}{g\sin\theta - \frac{b}{m}v^2}\ \mathrm{d}v = \int_0^X 1\ \mathrm{d}x$$

(Notice the limits – see the question on Jack and the cart for an explanation).

Using the tip in the question, we can do the integral on the LHS:

$$\left[-\frac{m}{2b}\ln\left|g\sin\theta - \frac{b}{m}v^2\right|\right]_{v_0}^{V} = X$$

$$\left[\ln\left|g\sin\theta - \frac{b}{m}v_0^2\right| - \ln\left|g\sin\theta - \frac{b}{m}V^2\right|\right] = \frac{2bX}{m}$$

$$\ln\left|\frac{mg\sin\theta - bv_0^2}{mg\sin\theta - bV^2}\right| = \frac{2bX}{m}$$

And so

$$\frac{mg\sin\theta - bv_0^2}{mg\sin\theta - bV^2} = e^{2bX/m}$$

$$mg\sin\theta - bV^2 = \left(mg\sin\theta - bv_0^2\right)e^{-2bX/m}$$

$$\boxed{V = \sqrt{\frac{mg\sin\theta - \left(mg\sin\theta - bv_0^2\right)e^{-2bX/m}}{b}}}$$

It is interesting (and reassuring) that as $X \to \infty$, we get

$$V \to \sqrt{\frac{mg\sin\theta}{b}}$$

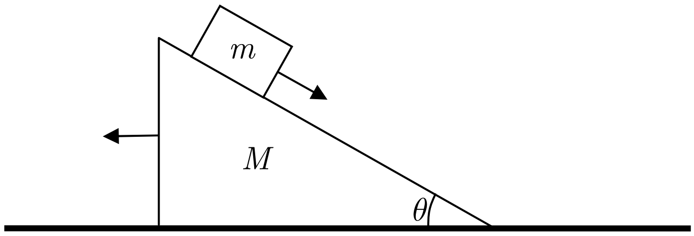
Which is the terminal velocity, and that at $X = 0$, we get

$$V = \sqrt{\frac{mg\sin\theta - \left(mg\sin\theta - bv_0^2\right)}{b}} = v_0$$

Which is indeed the starting velocity.

***** Question 5**

A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ . The plane rests on a frictionless horizontal surface. When the block is released, the block will start to move down the plane, but the plane will also move in the opposite direction, because it's not fixed! Calculate the horizontal acceleration of the plane:



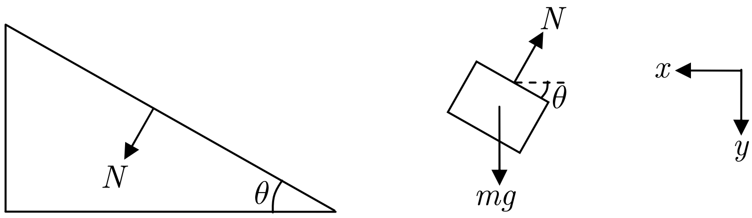
Solution:

This problem is **rock solid**! But if you follow the simple steps of dynamics, then it's very simple. It gave it to you to show you an example of a problem where the old A-Level way of doing things simply doesn't work, and you need the more rigorous and systematic framework of IA.

The question asks us for acceleration. There are four accelerations involved in this problem:

- a_x and a_y, the horizontal and vertical accelerations of the block.
- A_x, the horizontal acceleration of the plane, which we want to find
- A_y, the vertical acceleration of the plane, which is 0 (since the plane doesn't move up or down).

It's pretty clear we'll have to use $F = ma$, and to do that, we'll need free-body diagrams for the block and plane:



I've chosen to ignore the weight of the plane and the normal force of the table on the plane, because they clearly cancel out (since $A_y = 0$). Applying $F = ma$ these bodies, we get:

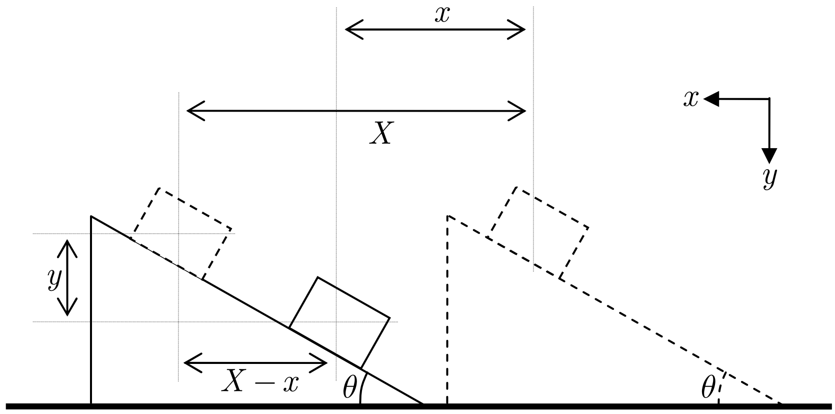
$$N\sin\theta = MA_x$$
$$mg - N\cos\theta = ma_y$$
$$-N\sin\theta = ma_x$$

At this point, the way forward isn't so clear. A **very useful method** in this kind of situation is to count the number of unknowns and the number of equations:

- 4 unknowns (N, A_x, a_y and a_x). The other variables don't count because they're allowed to appear in our final answer.
- 3 equations

Clearly, therefore, we're missing one equations – because to solve for n unknowns, we need n equations.

So let's go back to the problem – is there any physical aspect that we haven't modeled yet? The answer is yes – we haven't modeled the fact that the block needs to stay in contact with the plane at all times. How can we model that as an equation? Have a look at the diagram below:



Clearly, for the block to stay in contact, we need

$$\tan\theta = \frac{y}{X-x}$$
$$\left(X-x\right)\tan\theta = y$$

Differentiating this twice with respect to time, we get

$$\left(A_x - a_x\right)\tan\theta = a_y$$

So this is our last equation!

Let's summarize our equations so far:

$$
\begin{aligned}
N \sin\theta &= MA_x && (1)\\
mg - N\cos\theta &= ma_y && (2)\\
-N\sin\theta &= ma_x && (3)\\
\left(A_x - a_x\right)\tan\theta &= a_y && (4)
\end{aligned}
$$

We "simply" need to solve for A_x. Here are the boring details:

- Use (3) to eliminate N:

$$N = -\frac{ma_x}{\sin\theta}$$

  Which gives

$$
\begin{aligned}
-ma_x &= MA_x && (1)\\
g + \frac{a_x}{\tan\theta} &= a_y && (2)\\
\left(A_x - a_x\right)\tan\theta &= a_y && (4)
\end{aligned}
$$

- Use (4) to eliminate a_y, which gives

$$
\begin{aligned}
-ma_x &= MA_x && (1)\\
g + \frac{a_x}{\tan\theta} &= \left(A_x - a_x\right)\tan\theta && (2)
\end{aligned}
$$

- Use (2) to eliminate a_x. This requires some manipulation!

$$
\begin{aligned}
g + \frac{a_x}{\tan\theta} &= \left(A_x - a_x\right)\tan\theta\\
g\tan\theta + a_x &= A_x\tan^2\theta - a_x\tan^2\theta\\
a_x\left(1+\tan^2\theta\right) &= A_x\tan^2\theta - g\tan\theta\\
a_x\sec^2\theta &= A_x\tan^2\theta - g\tan\theta\\
a_x &= A_x\tan^2\theta\cos^2\theta - g\tan\theta\cos^2\theta\\
a_x &= A_x\sin^2\theta - g\sin\theta\cos\theta
\end{aligned}
$$

This finally gives

$$
\begin{aligned}
-m\left(A_x\sin^2\theta - g\sin\theta\cos\theta\right) &= MA_x\\
mg\sin\theta\cos\theta &= \left(M + m\sin^2\theta\right)A_x\\
\boxed{A_x = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}}
\end{aligned}
$$

***** Question 6 (loosely based on *Tripos 1995*)**

A crate of mass m is dropped vertically onto a rough conveyor belt that is moving at a speed v. A frictionless motor maintains the belt's constant speed through a frictionless mechanism. During the period in which the crate is being accelerated, find the work done by the motor which drives the belt.

Now consider a similar belt moving at a speed V on which mass is *continuously dropped* at a rate M unit mass per unit time. What is the *power* required to drive the motor. (This is the easier part of the question – you might want to do it first).

Compare your two answers. Isn't that interesting! Make sure you understand *why* this is the case.

Solution:

The first thought that might go through your head is "well, the block's kinetic energy goes from 0 to $\frac{1}{2}mv^2$, and so the work done by the belt is just $\frac{1}{2}mv^2$! You'd then look at the question, see the three stars next to it, and realize that maybe it's not that simple!

The problem with that simplistic reasoning is that energy is also dissipated as heat, as the crate slides over the belt before it is accelerated to a speed v. This means the work done by the motor is in fact greater.

I'll show you two methods to find the answer to this question. The first is methodical, the second is clever

First method – in the ground frame

You need to find the work done by the belt. It seems sensible, therefore, to find the total force the belt has to exert. This is just F_f, the force due to friction. We can therefore work out the work using

$$\text{Work} = F_f \times \text{Distance}$$

The distance over which the force acts is the distance the belt moves while the crate accelerates. We don't know what that is, but is there any way we can work it out? Indeed there is! The force is constant, so kinematics seems like a good idea.

If you guys, however, were thinking of finding out the distance the **block** travels, by assuming

$$a = F_f / m \qquad v_0 = 0 \qquad v_f = v \qquad x = ?$$

And then using $v_f^2 = v_0^2 + 2ax$ to get

$$\frac{mv^2}{2F_f} = x$$

You'd be **WRONG!!**

Why? Simply because in this case, we want the work done **by the belt**, and so we need the distance **the belt travels**. Since the belt always travels at v, whereas the block does not, the calculations above are bound to go screwy, because the belt doesn't travel the same distance as the block.

So, what do we do? After a few minutes of thought, you'll probably realize that you want to find the **time** the block takes to accelerate from 0 to v, and then simply multiply that by v to find the total distance traveled by the belt. To find the time, we use kinetics:

$$a = F_f / m \qquad v_0 = 0 \qquad v_f = v \qquad t = ?$$

Using $v_f = v_0 + at$, we get

$$t = \frac{mv}{F_f}$$

And so

$$\text{Distance travelled by belt} = v\frac{mv}{F_f} = \frac{mv^2}{F_f}$$

And so

$$\text{Work} = F_f \frac{mv^2}{F_f}$$

$$\boxed{\text{Work} = mv^2}$$

Interestingly, this is exactly double what we would have naively expected. The second method makes this transparent…

Second method – in the belt's frame

This method is more conceptually difficult, but algebraically much easier. The key insight is to note that the motor is doing two kinds of work here

1. It is accelerating the block, and giving it kinetic energy
2. It is dissipating energy to friction, as the block slips.

We can work out the amount of energy involved in each case

1. In the ground frame, the block starts at rest and eventually moves at speed v. The kinetic energy gained is $\boxed{\frac{1}{2}mv^2}$.
2. In the belt's frame, the block originally has energy v and then slows down to rest, with all the original energy dissipated to friction. In total, the energy lost to friction is $\boxed{\frac{1}{2}mv^2}$.

The total work down by the belt is then

$$W = \tfrac{1}{2}mv^2 + \tfrac{1}{2}mv^2 = mv^2$$

Precisely as above

Now let's do the continuous case. Surprisingly, it's much easier. We'll use the same strategy

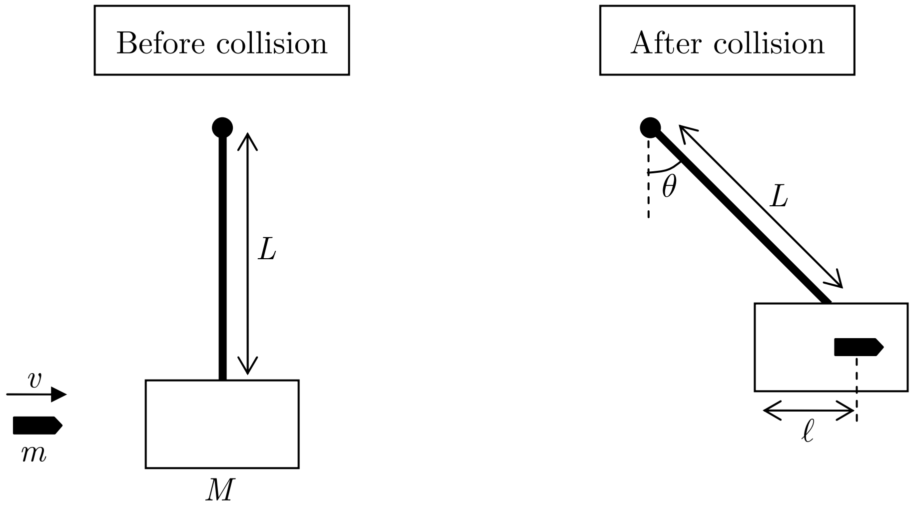
$$\text{Power} = \text{Force} \times \text{Velocity}$$

In this case, in an interval Δt, a mass $M\Delta t$ drops on the belt, and so the belt has to provide a momentum $MV\Delta t$. We know, however, that $F = \mathrm{d}p \,/\, \mathrm{d}t$, and so $F = MV$. Therefore

$$\boxed{\text{Power} = MV^2}$$

It is interesting that the result is precisely as above, simply because the second example is equivalent to dropping one crate of mass m onto the belt per second. Thus, the work done per second will obviously be equal to what we found above.

**** Question 7**



A block of mass M is attached to one end of a light rod of length L which is freely pivoted at the other end. A bullet of mass $m < M$ is shot into the block at a speed v and becomes embedded in the block. Assume that the bullet was moving so fast, and that the block is so hard, that the bullet becomes embedded instantly, even before the block starts moving. Calculate:

- The angle θ to the vertical to which the block rises before it comes to rest.
- Given that the force exerted by the wood on the bullet is F, find the distance ℓ traveled by the bullet *inside the block* before it stops.

Solution:

This is one of those problems in which the key is to realize that there are two parts to the problem.

1. **Inelastic collision of the block and the bullet [momentum conserved]**

   Let V be the final speed of the block/bullet and conserve momentum:

$$mv = \left(M + m\right)V$$

$$\boxed{V = \frac{mv}{M+m}}$$

2. **The block rises [energy conserved]**

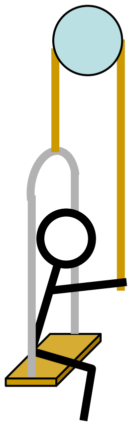
   As the block rises, it gains potential energy and loses kinetic energy. No work is done on the system, so the gain in GPE and the loss in KE are equal:

$$\frac{1}{2}\left(M+m\right)V^2 = mg\left(L - L\cos\theta\right)$$
$$\frac{m^2v^2}{2mg\left(M+m\right)} = L - L\cos\theta$$
$$\cos\theta = 1 - \frac{mv^2}{2gL\left(M+m\right)}$$
$$\boxed{\theta = \arccos\left\{1 - \frac{mv^2}{2gL\left(M+m\right)}\right\}}$$

To find how far the bullet gets embedded, we assume that *all* the energy lost in the inelastic collision is dissipated by the force F that resists the bullet as it digs itself into the block. As such

$$\text{Work done by } F = \text{KE lost}$$
$$F\ell = \frac{1}{2}mv^2 - \frac{1}{2}\left(M+m\right)V^2$$
$$F\ell = \frac{mv^2}{2} - \frac{m^2v^2}{2\left(M+m\right)}$$
$$\ell = \frac{1}{2F}\left(\frac{mv^2\left(M+m\right) - m^2v^2}{\left(M+m\right)}\right)$$
$$\boxed{\ell = \frac{Mmv^2}{2F\left(M+m\right)}}$$

*** Question 8**

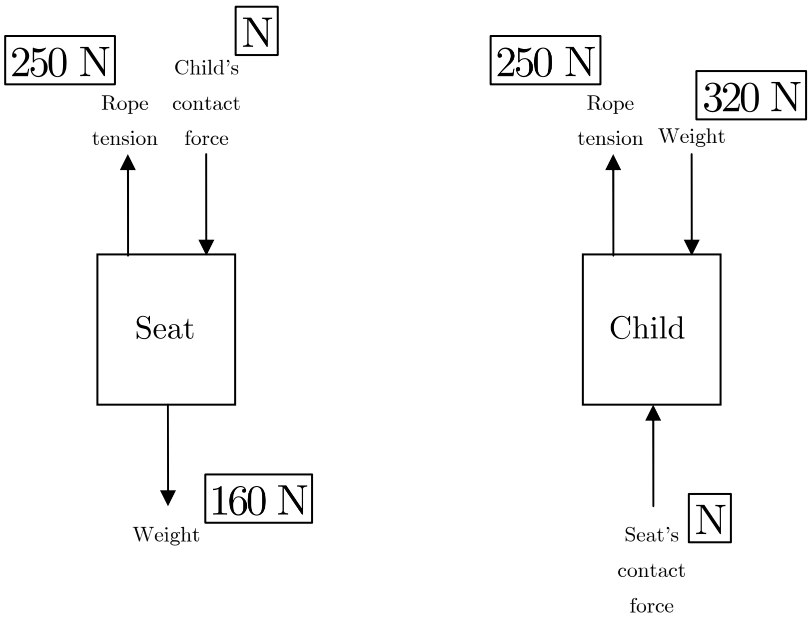


A child is sitting in a chair connected to a rope that passes over a frictionless pulley. The child pulls on the loose end of the rope with a force of 250 N. The child's weight is 320 N and the chair weighs 160 N. The child is accelerating. Find the force that the seat of the chair exerts on the child.

Answer: 80 N

Solution:

Let's draw some free-body diagrams:



(Note that the rope tension is the same in both parts of the rope because the pulley is perfect).

The child always stays in contact with the seat, so the child and seat must be accelerating at the same rate. Thus, the resultant force on each body must be the same. Taking upwards as positive, we get:

$$250 - N - 160 = 250 - 320 + N$$
$$160 = 2N$$
$$\boxed{N = 80 \text{ N}}$$

**** Question 9**

Jack sits on a cart loaded with stones and free to slide on a frictionless road. The total mass of Jack, the cart and the stones is m_0. Jack propels the cart by throwing stones out of the back of the cart at a constant speed u relative to the cart, and at a constant rate μ (mass per unit time). Jill helps along by pushing the cart with a constant force F. What is the speed of the cart at a time t after Jack and Jill have started pushing it?

Solution:

This is clearly a case of lots of collisions happening continuously in time. Let's consider the collision that occurs during the time interval Δt. Before we do that, two preliminary points:

- In those problems, the following rule of thumb will rarely fail you:

  Whenever the velocity of the ejected material is given
  with respect to the rocket/cart/etc…
  Then
  It (almost) ***always*** **makes the algebra easier to deal**
  with the situation in the rest frame of the
  rocket/cart/etc… at the start of the interval.

  So instead of having the cart go from v to $v + \Delta v$, we'll have it go from 0 to Δv.

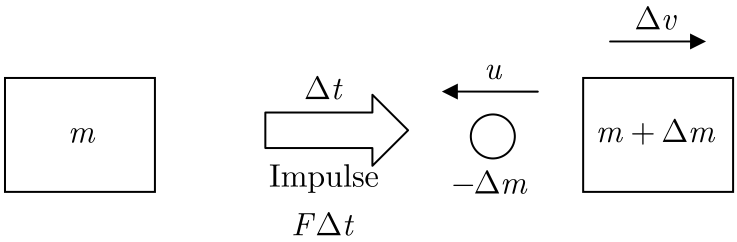
- It is **CRUCIAL** to carefully consider what dm represents here[3]. m is the mass of the cart. And so dm has to be **the increase in mass of the cart**. This means that

  You should always make the mass of the cart
  go from ***m*** **before the interval** to ***m*** **+ d*m***
  after the interval. If the mass of the cart is
  actually decreasing, **d*m*** **will end up negative**;
  that's OK!

[3] How to deal with the sign of dm is one of those very few points which none of my supervisors could clarify for me when I was in IA. It took me three years to finally figure it out, and it only happened as I was preparing to lecture the topic in MIT… Teaching really is the best way to learn something properly!

> If you choose some other convention (for example, in this case, where the rocket is losing mass, you might be tempted to make it go from $m + \mathrm{d}m \Rightarrow m$) you'll end up getting the signs dreadfully messed up

So, after that lengthy introduction, let's do our drawing:



[Notice how the cart goes from m to $m + \Delta m$. That's OK – the ejected item just needs to have mass $-\Delta m$ to conserve mass, and Δm will end up being negative].

Clearly, momentum isn't conserved – but we know the impulse, and so we can write (ignoring the product of infinitesimal quantities):

$$\text{Impulse} = \Delta\text{Momentum}$$
$$F\Delta t = \left(m + \Delta m\right)\Delta v - \left(-\Delta m\right)u - 0$$
$$F\Delta t = m\Delta v + u\Delta m$$

Committing the mathematical blasphemy of changing our small quantities to infinitesimal ones and dividing by $\mathrm{d}t$, we get

$$F = m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t}$$

We want v in terms of t, so we need to get rid of any appearances of m in this equation. We do that by noting that:

- $\mathrm{d}m / \mathrm{d}t = -\mu$ (remember, m is the mass of the cart, so the derivative is negative, because it's decreasing).
- $m = m_0 - \mu t$

We then get

$$F = \left(m_0 - \mu t\right)\frac{\mathrm{d}v}{\mathrm{d}t} - u\mu$$

Let's get the derivative by itself:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{F + u\mu}{m_0 - \mu t}$$

Now, we simply need to integrate with respect to time:

$$\int \frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t = \int \frac{F+u\mu}{m_0-\mu t}\ \mathrm{d}t$$
$$\int \mathrm{d}v = \int \frac{F+u\mu}{m_0-\mu t}\ \mathrm{d}t$$

How do we choose the limits on our integral? Use the following simple rule:

v and t at the **end** of the

motion considered

$$\int_{\square}^{\square}\mathrm{d}v = \int_{\square}^{\square}\mathrm{d}t$$

v and t at the **start** of the

motion considered

In this case, we're considering motion from the cart at rest at a time 0 ($v = 0$ and $t = 0$), to the cart moving at a speed v at a time t. So:

$$\int_0^v \mathrm{d}v = \int_0^t \frac{F+u\mu}{m_0-\mu t}\ \mathrm{d}t$$
$$v = \left(F+u\mu\right)\left[-\frac{1}{\mu}\ln\left|m_0-\mu t\right|\right]_0^t$$
$$v = \left(F+u\mu\right)\left[-\frac{1}{\mu}\ln\left|m_0-\mu t\right| + \frac{1}{\mu}\ln\left|m_0\right|\right]$$
$$\boxed{v = \frac{F+u\mu}{\mu}\ln\left(\frac{m_0}{m_0-\mu t}\right) \qquad 0 \le t < \frac{m_0}{\mu}}$$

[Do you understand why I've removed the absolute signs in the last step, and added the constraint on t? Also, what happens when $t = m_0 / \mu$? The equation predicts something terrible will happen, but in practice, what does that mean? Is the model actually valid for **all** $t < m_0 / \mu$?]

The only other type of question possible here would be to find v in terms of m. Make sure you know how to do that.

[*Note for the masochists amongst you: there is also a way to find the differential equation directly from* $F = \mathrm{d}p / \mathrm{d}t$ *. There'll be a £1 cheque from me for anyone who manages it!*]

*** Question 10 (*Tripos 2004*)**

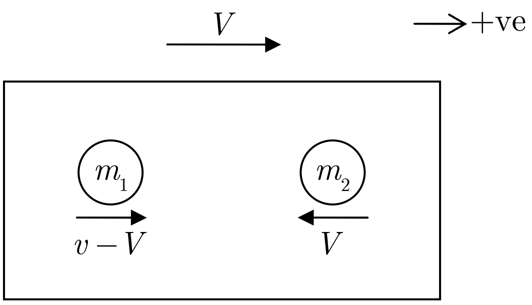
A particle of mass m_1 and velocity v makes an elastic, head-on collision with a stationary particle of mass m_2. Find the velocity of the zero momentum frame relative to the laboratory frame. By considering the collision in the zero-momentum frame, show that in the laboratory frame, the fraction of the initial kinetic energy transferred to m_2 is given by

$$\frac{4m_1m_2}{\left(m_1+m_2\right)^2}$$

Three balls of masses m_1, m_2 and m_3 are suspended in a horizontal line by light wires and are almost touching. The mass m_1 is given a horizontal velocity v so that it collides head-on with the mass m_2. Find an expression for the final kinetic energy of m_3 and sketch it as a function of m_2. What value of m_2 results in the maximum energy transfer to the mass m_3?

Solution:

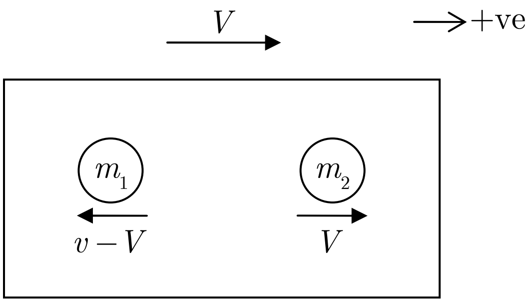
Imagine the zero-momentum frame is moving at a speed V to the right. The situation in that frame looks as follows (note that in these problems, where we don't know any of the quantities, it is even more crucial than usual to indicate what direction we choose to be positive):



We want the momentum in this frame to be 0. As such

$$m_1\left(v-V\right)=m_2V$$
$$m_1v-m_1V=m_2V$$
$$\boxed{V=\frac{m_1v}{m_1+m_2}}$$

Consider the situation in the ZMF after the collision. It's pretty clear that the particles *must* both reverse their directions (to keep the momentum 0)[4]. And here comes the nifty trick[5]; I claim that in the ZMF after the collision the particles **must** have the **same speeds** as they did before, but juts reversed in direction. In other words, I claim that after the collision, the situation is as follows:

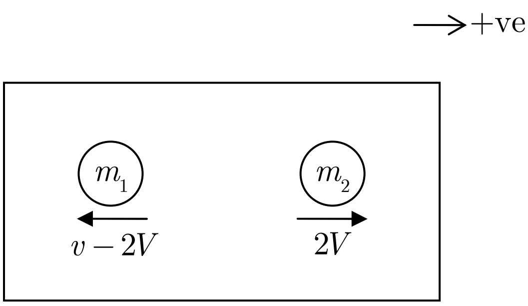


Why must that be the case? Consider the following argument:

- To keep the momentum 0, the ratio of the speeds **after** the collision must be the **same** as they were **before** the collision.
- This means that they must either **both increase**, **both decrease** or **both stay the same**.
- However, the first two options are impossible, because that would create or destroy energy, and we are told the collision is elastic.

Thus, the situation *must* be as we depicted it in the ZMF.

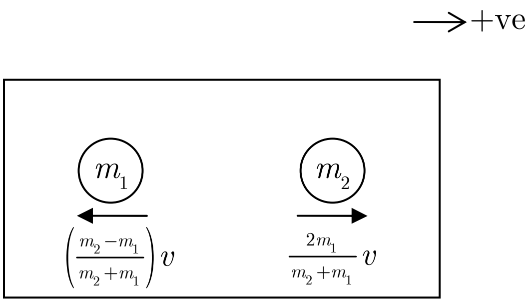
Translating back to the lab frame, we get:



Substituting in our expression for V, we get

[4] If, however, you *hadn't* realised that, and you'd made the wrong assumption, no problem. It just means you would end up with an extra minus sign at the end of your calculation. That minus sign in front of one of the velocities would tell you that your assumption was wrong.

[5] For you guys who actually tried to conserve momentum and energy manually, I sympathise – I also wasted many hours doing that at some point in my life. Once you've done it once, though, you'll never make that mistake again!



Finally, let's find the original kinetic energy and the kinetic energy transferred to m_2

$$\mathrm{KE}_{\text{original}} = \frac{1}{2}m_1v^2$$

$$\mathrm{KE}_{m_2} = \frac{1}{2}m_2v^2\left(\frac{2m_1}{m_2+m_1}\right)^2$$

The fraction transferred is

$$\mathrm{Frac} = \frac{\mathrm{KE}_{m_2}}{\mathrm{KE}_{\text{original}}} = \frac{m_2}{m_1}\frac{4m_1^2}{\left(m_1+m_2\right)^2}$$

$$\boxed{\mathrm{Frac} = \frac{4m_1m_2}{\left(m_1+m_2\right)^2}}$$

As requested.

The next question is reasonably simple given this result. We note that the original kinetic energy given to mass m_1 is $m_1u^2/2$. Thus, the kinetic energy transferred to mass m_2 is

$$\frac{1}{2}m_1u^2\frac{4m_1m_2}{\left(m_1+m_2\right)^2}$$

And of that, the energy transferred to m_3 is

$$\boxed{\mathrm{KE}_{m_3} = \frac{1}{2}m_1u^2\frac{4m_1m_2}{\left(m_1+m_2\right)^2}\frac{4m_2m_3}{\left(m_2+m_3\right)^2}}$$

In terms of sketching this as a function of m_2, it helps to re-write it as

$$\begin{aligned}
\mathrm{KE}_{m_3} &= 8u^2m_1^2m_3\frac{m_2^2}{\left(m_1+m_2\right)^2\left(m_2+m_3\right)^2} \\
&= 8u^2m_1^2m_3m_2^2\left(m_1m_2+m_2^2+m_1m_3+m_2m_3\right)^{-2} \\
&= 8u^2m_1^2m_3\left(m_1+m_2+m_3+\frac{m_1m_3}{m_2}\right)^{-2}
\end{aligned}$$

Plotting this seems like a daunting task, so let's do it step by step:

- Clearly, it's positive for all m_2
- As $m_2 \to 0$, $\mathrm{KE}_{m_3} \to 0$ (do you understand why I didn't write "=")? To see why this is true, look at the original expression for KE_{m_3}.
- As $m_2 \to \infty$, $\mathrm{KE}_{m_3} \to 0$ (because the term inside the bracket goes to infinity)

So it looks like the graph starts at 0, rises up to a maximum, and then falls again down to 0.

Finding the derivative here is **not** a good idea, because it's absolutely ghastly. You're much better of realizing that to maximize the entire function, all you really want to do is minimize the stuff in the brackets – in other words, you want to **minimize**

$$m_1 + m_2 + m_3 + \frac{m_1 m_3}{m_2}$$

Differentiate and set to 0

$$1 - \frac{m_1 m_3}{m_2^2} = 0$$

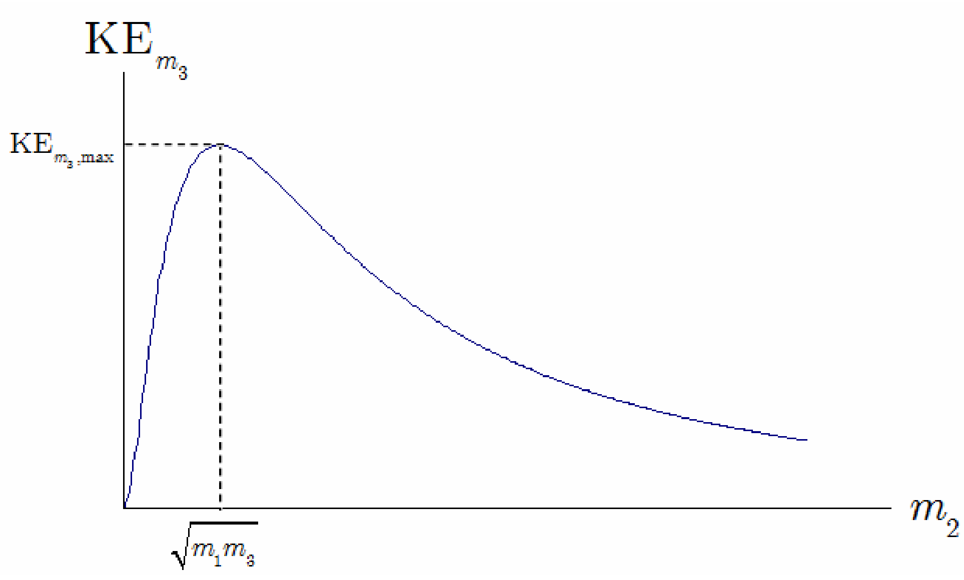
$$\boxed{m_2 = \sqrt{m_1 m_3}}$$

At that point, the function has the value:

$$\mathrm{KE}_{m_3,\max} = 8u^2 m_1^2 m_3 \left(m_1 + m_3 + \sqrt{m_1 m_3} + \frac{m_1 m_3}{\sqrt{m_1 m_3}} \right)^{-2}$$

$$\boxed{\mathrm{KE}_{m_3,\max} = 8u^2 m_1^2 m_3 \left(m_1 + m_3 + 2\sqrt{m_1 m_3} \right)^{-2}}$$

And the graph looks like:


KE_{m_3}
$KE_{m_3,\max}$
m_2
$\sqrt{m_1 m_3}$

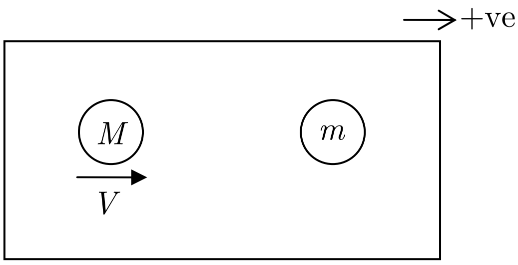
*** Question 11

A sheet of mass M moves with a speed V (in the direction of its normal) through a region of space that contains very light stationary particles of mass $m \ll M$. There are n of these particles per unit volume. All collisions are elastic.

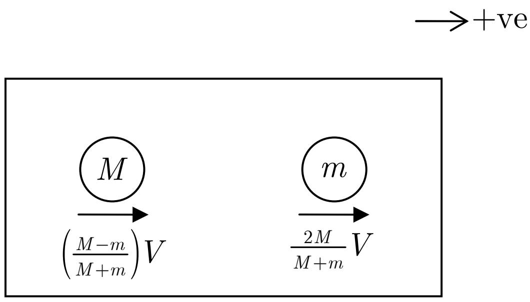
What is the drag force per unit area on the sheet?

Solution:

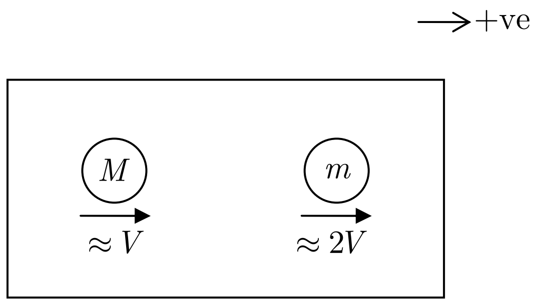
The first thing we need to do is to consider the collision of the wall with *one* of these stationary molecules. The situation in the lab frame is as follows:



The eagle-eyed amongst you will have noticed that this is exactly the situation we had in the previous question. Plagarising liberally, we have that the situation in the lab frame after the collision will be:



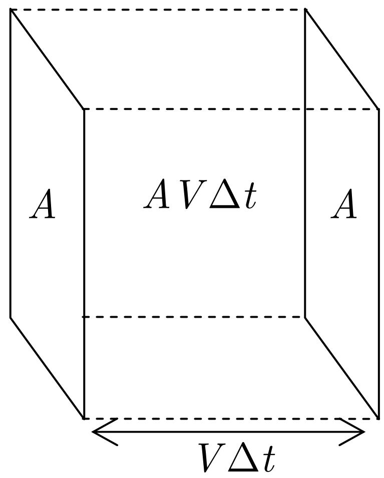
However, in this question, we have that $m \ll M$ (in fact, it'd be more accurate to say $m \lll\lll M$ – indeed, given that we're trying to model air resistance, m would be the mass of an oxygen molecule whereas M would be the mass of, say, a rocket!) Thus, we can assume $m \approx 0$, and get:



So effectively, the effect of the collision has been to give the stationary particle a "bumph" in momentum of magnitude $2mV$. That momentum, of course, can't come from no-where… So we see that as a result of each collision, we must supply the plane with extra momentum $2mV$ to ensure it can remain moving at a speed V.

Now, the next stage in this argument is **crucial** in kinetic theory, which you'll come across in IB (to my great dismay, it looks like it's been removed from IA). If you understand this argument now, it'll make your life infinitely easier when you come across is later.

Consider a time interval Δt. In that time, the plane will travel a distance $V\Delta t$, and will therefore "sweep out" a volume $AV\Delta t$:



The question tells us, though, that there are n of these particles per unit volume. This means that in a time Δt, as the plane sweeps through $AV\Delta t$, the number of particles it encounters is $nAV\Delta t$. However, we saw above that we need to supply momentum $2mV$ for each of these collisions. Thus the total momentum that needs to be supplied in the time Δt is

$$2mnAV^2\Delta t$$

Where does that momentum come from? Clearly, from a force on the plane. Where does that force come from? Well, we know that $F = \mathrm{d}p \,/\, \mathrm{d}t$, and so

$$F = 2mnAV^2$$

And of course, this is precisely the drag force we were looking for! So per unit area, the drag force is

$$\boxed{\frac{F}{A} = 2mnV^2}$$

This should convince you that I wasn't completely and utterly mad when I told you, earlier on, that the drag force on a very fast moving block was proportional to the velocity of the block squared! This derivation, of course, is only valid when the plane is moving much faster than the air particles, because we assumed the air particles were at rest…

Just for the sake of giving you some practice with impulse, I just want to show you how to solve it slightly differently. We know that

$$\text{Impulse} = \text{Change in momentum}$$

We therefore know that over the time Δt, the impulse on the plane must be

$$I = 2mnAV^2\Delta t$$

However, we also know that

$$I = \int F \,\mathrm{d}t \Rightarrow F = \frac{\mathrm{d}I}{\mathrm{d}t}$$

It therefore is pretty clear, once again, that

$$F = 2mnAV^2$$