St John's College

Physics Example Classes

CLASS 3 – WAVES & QUANTUM WAVES

GENERAL NOTES

Schrodinger's Equation

The stationary wavefunctions of a particle of mass m and energy E in a potential V always have wavefunction

$$\Psi(x,t) = e^{-i\omega t}\psi(x)$$

where $\psi(x)$ must satisfy the Schrodinger Equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi$$
$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

There's often a lot of conclusion as to how to deal with this, so here's a concise summary.

What makes this equation so easy to solve in IA is that in all the cases you'll be given, V is constant. This means that the equation above is a second order differential equation with constant coefficients, which you know how to solve. We re-write the equation as

$$\frac{\partial^2\psi}{\partial x^2} + k^2\psi = 0 \qquad \boxed{k^2 = \frac{2m}{\hbar^2}(E-V)}$$

The auxiliary quadratic is

$$\lambda^2 + k^2 = 0$$
$$\lambda = \pm ik$$

We then have two options

$$\boxed{\boxed{k \text{ real}}}$$

This occurs if $E > V$ (ie: the particle is classically able to exist in the region). The general solution is then

$$\psi(x) = \alpha e^{ikx} + \beta e^{-ikx}$$
$$\boxed{\psi(x) = A\cos(kx) + B\sin(kx)}$$

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$\boxed{k \text{ imaginary}}$

This occur if $E < V$ (ie: the presence of the particle in this region is classically forbidden) and the general solution is then

$$\boxed{\psi(x) = Ae^{kx} + Be^{-kx}}$$

The constants A and B can be found by using boundary conditions. These are:

- The wavefunction must be continuous
- The wavefunction must be 0 wherever $V = \infty$
- [The wavefunction must be differentiable at all points, unless a jump to or from $V = \infty$ is occurring] – I think you're less likely to use this one in IA.

Note that even though the full form of the wavefunction is $\Psi(x,t) = e^{-i\omega t}\psi(x)$, the time part is often ignored, and we just consider $\psi(x)$. The reason is that the true expression for the probability density of the particle is (assuming $\psi(x)$ is real)

$$\Psi(x,t)\Psi^*(x,t) = e^{-i\omega t}\psi(x)e^{i\omega t}\psi^*(x) = \psi(x)\psi^*(x) = \left|\psi(x)\right|^2$$

The time part simply vanishes. Another way to think about it is that when we make an observation, we're "averaging over time" and so the precise value of the wavefunction at that point in time doesn't matter.

Waves

People often get very confused by the many terms that float around when we talk about waves: wavelength, wavenumber, frequency, angular frequency, period and speed. I want to give a succinct summary of what each of those terms mean, and how they relate to each other.

General Waves

We first consider a general wave (as opposed to a sinusoidal one) which is basically just a repeating pattern moving in space. There are two important properties of such a wave

<table>
<tr><th>WAVELENGTH λ</th><th>FREQUENCY ν (OR PERIOD T)</th></tr>
<tr><td>Tells you something about the way the wave propagates through <u>space</u></td><td>Tells you something about the way the wave propagates through <u>time</u></td></tr>
<tr><td>The wavelength λ is the distance between two identical points on a wave, at any given time.

[ie: freeze time and see how the wave propagates in space]</td><td>The period T is the time it takes for a full wave to pass a given point. It's the exact analogue of the wavelength in time.

The frequency ν is the number of full waves that pass any given point every second.

ν and T are related by
$$\nu = 1/T$$
[ie: freeze space by staying at one point, and see how the wave propagates in time]</td></tr>
</table>

Finally, there is a third quantity that **links** the space and time properties of the wave, and that is the **velocity** c:

$$\text{Velocity} = \text{Frequency} \times \text{Wavelength}$$

$$\boxed{c = \nu\lambda}$$

So given any two of c, ν and λ, you can find the third one.

Sinusoidal Waves

We now move on to sinusoidal waves, of the form

$$\boxed{\sin\left(\omega t - kx\right)}$$

The quantities ω and k are called the **angular frequency** and the **wavenumber**. The key point I want you to realise, however, is that they're just another way to measure the two quantities we've seen before:

Wavenumber k	**Angular Frequency ω**
Describes how the wave propagates through space	*Describes how the wave propagates through time*
To relate the **wavelength λ** to the **wavenumber k**, simply note that if $k = 1$, the wavelength of sin would be 2π. Increasing k shrinks this wavelength. Thus $\boxed{\lambda = \frac{2\pi}{k}} \Rightarrow \boxed{k = \frac{2\pi}{\lambda}}$	To relates the **frequency ν** to the **period T**, simply note that if $\omega = 1$, the period of sin would be 2π. Increasing ω shrinks the period, and so $\boxed{T = \frac{2\pi}{\omega}} \Rightarrow \boxed{\omega = \frac{2\pi}{T}}$ And using $\nu = 1/T$, we get $\boxed{\nu = \frac{\omega}{2\pi}} \Rightarrow \boxed{\omega = 2\pi\nu}$

Similarly, using $c = \nu\lambda$, we get

$$c = \frac{\omega}{2\pi}\frac{2\pi}{k} = \frac{\omega}{k}$$

It's important to remember that these two quantities (ω and k) are simply a way of **rescaling** λ and ν. They don't describe any new physical quantities!

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SOLUTIONS

Question 1 – Shorties

(a) (*Based on Tripos 2006*) A particle of mass m is confined in a one-dimensional potential having

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{elsewhere} \end{cases}$$

What is the energy of the particle when it is in its n^{th} energy level? Calculate the probability that the particle is found in the region $\frac{1}{4}a < x < \frac{3}{4}a$ when the particle is in its n^{th} energy level. What does this result tend to as $n \to \infty$? Is this what you would expect classically?

(b) (*Tripos 2005*) A camera has a lens with focal length 3cm. Over what distance must the lens be movable to allow objects between 0.7m and infinity from the lens to be imaged in focus at a fixed plane?

(c) (*Based on Tripos 2009*) Find the two lowest energy levels of an electron sitting in a box of dimensions $a \times b \times c$. What is the degeneracy of each of these levels? [Remember that some or all of the dimensions might be equal!]

(d) (*Tripos 2004*) A square well potential has

$$V(x) = \begin{cases} -V_0 & |x| < a \\ 0 & \text{elsewhere} \end{cases}$$

with $V_0 > 0$. What are the boundary conditions that the wavefunction must satisfy at $x = -a$ and $x = a$? If V_0 is sufficiently large that there are many bound states, sketch the wavefunctions of each of the three lowest energy levels.

(e) A short-wave (HF) radio receiver receives simultaneously two signals from a transmitted D away, one by a path along the surface of the earth, and one by reflection from a portion of the ionospheric layer situated at a height of H. The layer acts as a perfect horizontal reflector. When the frequency of the transmitted wave is ν , it is observed that the combined signals strength varies from maximum to minimum and back to a maximum N times per minute. With what slow vertical speed is the ionosphoric layer moving? (Assume the earth is flat, and ignore atmospheric disturbances).

(f) A uniform inextensible string of length ℓ and total mass M is suspended vertically and tapped at the top end so that a transverse impulse runs down it. At the same moment, a body is released from rest and falls freely from the top of the string. How far from the bottom does the body pass the impulse?

(g) What is the mean power required to maintain a travelling wave of amplitude y_0 and wavelength λ on a string of mass per unit length ρ and under tension T?

(h) A converging lens forms a real image of an object. Show that there are two possible positions for the lens that will lead to the *same* distance between the object and the imagine, and that the size of the object is given by $\sqrt{h_1 h_2}$, where h_1 and h_2 are the sizes of the two images.

(i) Define the following terms, and their relationship to each other in the context of waves: wavelength, wavenumber, wave-vector, frequency, angular frequency, period, amplitude and speed.

Solutions:

(a) Using the hints in the introductory section of these solutions, we know the wavefunction will be 0 outside the well. Inside the well, $E > V$, since $E > 0$, and so the wavefunction will have the form

$$\Psi(x,t) = \left\{A\cos(kx) + B\sin(kx)\right\}e^{-i\omega t}$$

From now on, we ignore time dependence, because everything we observe is averaged over time anyway:

$$\psi(x) = A\cos(kx) + B\sin(kx)$$

The wavefunction needs to be continuous at $x = 0$ and $x = a$. Thus:

$$\psi(0) = Ae^{-i\omega t} = 0 \Rightarrow A = 0$$

$$\psi(a) = B\sin(ka) = 0 \Rightarrow k = \frac{n\pi}{a}$$

To find B, we need to ensure our wavefunction is normalized:

$$\int_{-\infty}^{\infty} \left|\psi(x)\right|^2 \, \mathrm{d}x = 1$$

$$\int_0^a B^2 \sin^2(n\pi x / a) \, \mathrm{d}x = 1$$

$$\frac{B^2}{2}\int_0^a 1 - \cos\left(\tfrac{2n\pi}{a}x\right) \, \mathrm{d}x = 1$$

$$\frac{B^2}{2}\left[x - \tfrac{a}{2n\pi}\sin\left(\tfrac{2n\pi}{a}x\right)\right]_0^a = 1$$

$$\frac{B^2}{2}\left[a - \tfrac{a}{2n\pi}\sin\left(2n\pi\right) - 0 + \tfrac{a}{2n\pi}\sin\left(0\right)\right] = 1$$

$$B = \sqrt{\frac{2}{a}}$$

And so the wavefunction of the particle when it is in its n^{th} energy level is

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)}$$

To find the energy of the particle when it is in its n^{th} energy level, remember that $k_n = n\pi / a$, $k^2 = \frac{2m}{\hbar^2}(E - V)$ and that in this case, $V = 0$. Therefore, in this case

$$\frac{n^2\pi^2}{a^2} = \frac{2m}{\hbar^2}E$$

$$\boxed{E = \frac{\hbar^2 n^2 \pi^2}{2ma^2}}$$

If the particle is in its n^{th} energy level, the probability it will be found in the region $\frac{1}{4}a < x < \frac{3}{4}a$ is

$$P(\tfrac{a}{4} \le x \le \tfrac{3a}{4}) = \int_{a/4}^{3a/4} \left|\psi(x)\right|^2 \, \mathrm{d}x$$

We've already worked out this integral above, so we can jump straight to the solution:

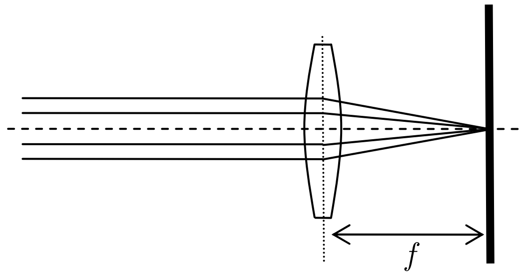
$$\begin{aligned} P(\tfrac{a}{4} \le x \le \tfrac{3a}{4}) &= \frac{1}{a}\left[x - \tfrac{a}{2n\pi}\sin\left(\tfrac{2n\pi}{a}x\right)\right]_{a/4}^{3a/4} \\ &= \frac{1}{a}\left[\frac{3a}{4} - \frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}\frac{3a}{4}\right) - \frac{a}{4} + \frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}\frac{a}{4}\right)\right] \\ &= \frac{1}{2} + \frac{1}{2n\pi}\left\{\sin(n\pi / 2) - \sin(3n\pi / 2)\right\} \end{aligned}$$

Using the identity $\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$, we obtain

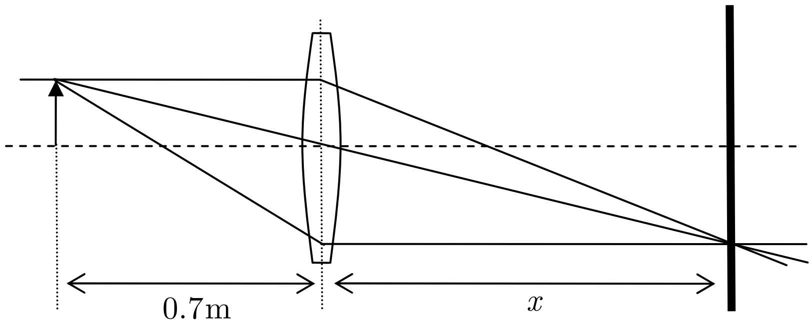
$$P(\tfrac{a}{4} \le x \le \tfrac{3a}{4}) = \frac{1}{2} - \frac{\sin\left(n\frac{\pi}{2}\right)\cos\left(n\pi\right)}{n\pi}$$

Clearly, as $n \to \infty$, $P(\tfrac{a}{4} \le x \le \tfrac{3a}{4}) \to 1/2$. This is precisely what we would expect in the classical limit, because the region $\frac{1}{4}a < x < \frac{3}{4}a$ occupies about half the well, so we would expect the particle to be there roughly half the time.

(b) Let's begin with a picture (as usual!). Consider rays coming in from infinity:



Now consider rays coming in from 0.7m away:



We can find x as follows:

$$\frac{1}{0.7} + \frac{1}{x} = \frac{1}{f}$$
$$x = 0.031 \text{ m} = 3.13 \text{ cm}$$

So if we want both images to be focused on a fixed screen (the thick line above), then the lens needs to be able to be moved a distance

$$\boxed{x - f = 1.3 \text{ mm}}$$

(c) First, we assume without loss of generality that $a \le b \le c$. If that's not the case, we can re-order them (or rotate our box) to make it the case.

Now, we quote from part (a) the result that the energy levels of a particle of mass m in a *one dimensional* box of width a are

$$E = \Delta \frac{n^2}{a^2} \qquad \Delta = \frac{\hbar^2 \pi^2}{2m}$$

The box in this case has three dimensions, and each of those dimensions will contribute a similar energy to the total:

$$E_a = \Delta \frac{n_a^2}{a^2} \qquad E_b = \Delta \frac{n_b^2}{b^2} \qquad E_c = \Delta \frac{n_c^2}{c^2}$$

With

$$E_{\text{total}} = E_a + E_b + E_c$$

Clearly, the lowest energy level occurs when $n_a = n_b = n_c = 1$, in which case

$$E_1 = \Delta \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)$$

The degeneracy of this level is 1, because there's only one way to get that energy.

The next level up will obviously involve increasing one of the n from 1 to 2. Since $c \geq b \geq a$, $1/c^2$ is the smallest term in the expression aobve. Thus, we'll set $n_z = 2$, and get

$$E_2 = \Delta \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{4}{c^2} \right)$$

The degeneracy is a bit more complicated to work out:

- If $a \neq b \neq c$, then the only way to get E_2 is as above, and the degeneracy is therefore 1.
- If $b = c \neq a$, then we can obtain E_2 as above, but we can also obtain it as follows

  $$E_2 = \Delta \left(\frac{1}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} \right)$$

  And so the degeneracy is 2.
- If $a = b = c$, then we can obtain E_2 in the two ways above, and also as follows

  $$E_2 = \Delta \left(\frac{4}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)$$

  and so the degeneracy is 3.

When we try to find the third energy level, all hell breaks loose. Let's, once again, consider three cases:

- If $a \neq b \neq c$, recall that E_2 was

$$E_2 = \Delta\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{4}{c^2}\right)$$

we now have three choices

(1) Add an extra quantum to n_c (ie: make $n_c = 3$), which adds energy $5\Delta / c^2$.

(2) Add an extra quantum to n_b (ie: make $n_b = 2$) which adds energy $3\Delta / b^2$

(3) Add an extra quantum to n_a (ie: make $n_a = 2$) which adds energy $3\Delta / a^2$

Clearly, choice (3) is no good, because choice (2) will always involve the addition of a smaller energy (since $a < b$). Whether you choose (1) or (2) depends on how big c is compared to b. In fact, if

$$\frac{5\Delta}{c^2} < \frac{3\Delta}{b^2} \Rightarrow b < \sqrt{\frac{3}{5}}c$$

then we should choose (1). Otherwise, choose (2).

Either way, there's only one way to do it, so the degeneracy is 1.

- If $a \neq b = c$, recall that E_2 was one of the following

$$E_2 = \Delta\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{4}{c^2}\right) = \Delta\left(\frac{1}{a^2} + \frac{4}{b^2} + \frac{1}{c^2}\right)$$

Once again, we have three choices

(1) Add an extra quantum to the n that already has a quantum in it, which adds energy $5\Delta / b^2 = 5\Delta / c^2$.

(2) Add an extra quantum to the n that *doesn't yet* have an extra quantum in it, which adds an energy $3\Delta / b^2 = 3\Delta / c^2$.

(3) Add an extra quantum to n_a (ie: make $n_a = 2$) which adds an energy $3\Delta / a^2$

Clearly, choice (3) is no good, because choice (2) will always involve the addition of a smaller energy (since $b < a$). Furthermore, it's clear that choice (2) will always result in the addition of less energy than choice (1). Thus, choice (2) is the only one left.

The degeneracy in this case is simply 1, because whichever one of the E_2 you choose, once you've performed choice (2) on it, you'll always end up with the same state.

- If $a = b = c$, recall that E_2 was one of the following

$$\begin{aligned} E_2 &= \Delta\left(\frac{4}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \\ &= \Delta\left(\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}\right) \\ &= \Delta\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{4}{c^2}\right) \end{aligned}$$

To get our third energy level, we have two choices:

(1) Add an extra quantum to the n that already has a quantum in it, which adds an energy $5\Delta / a^2 = 5\Delta / b^2 = 5\Delta / c^2$.

(2) Add an extra quantum to an n that doesn't yet have a quantum in it, which adds an energy $3\Delta / a^2 = 3\Delta / b^2 = 3\Delta / c^2$

Clearly, the second option leads to a small energy.

If you apply (2) to each of the above, you'll find that you only end up with 3 different configurations. Thus, the degeneracy is 3.

(d) If we have a "bound state", it means that the particle cannot escape the well. The energy of the particle in such a state is therefore smaller than 0. Therefore:

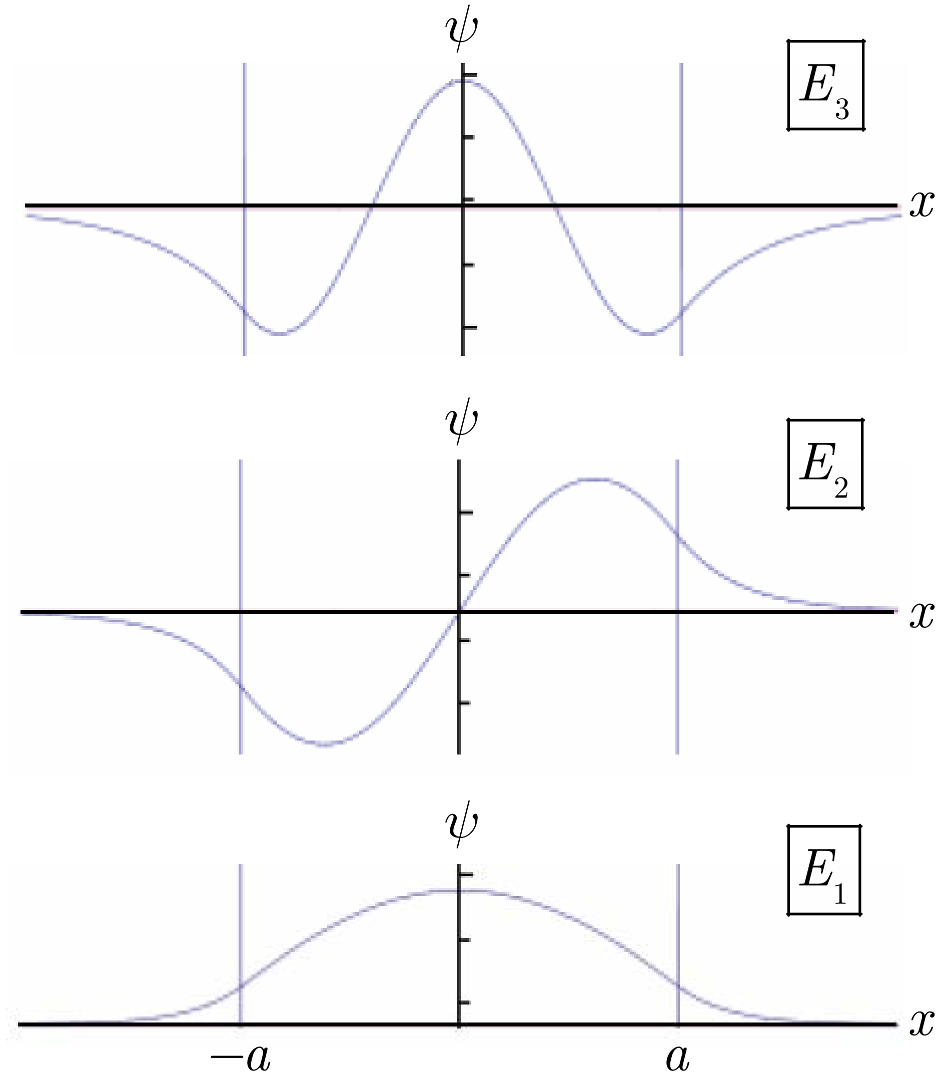
- Outside the well, the wavefunction will be an evanescent exponential
- Inside the well, the wavefunction will be sinusoidal.

The boundary conditions the wavefunction has to satisfy at the edge of the well are:

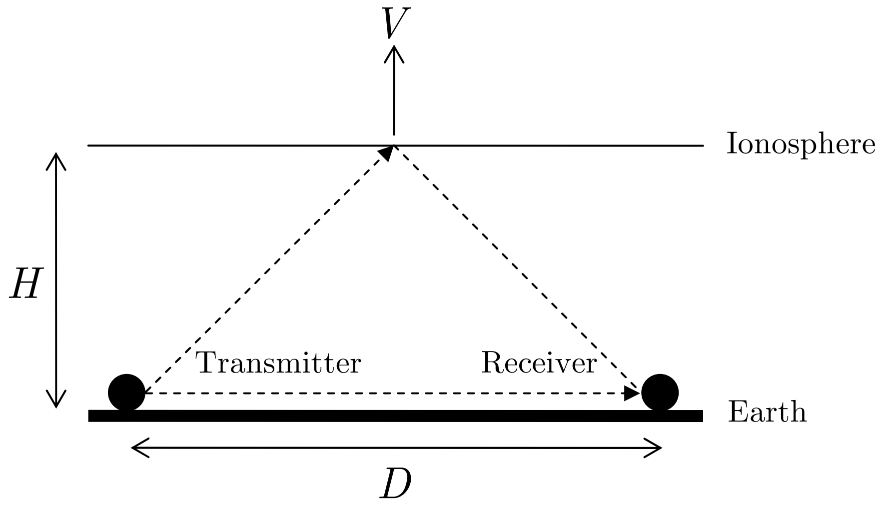
- The wavefunction has to be continuous

- The derivative of the wavefunction has to be continuous (since the potential is not jumping to infinity).

The wavefunctions of the three lowest energy levels will therefore look something like this:



(e) Let's first summarize the information in the question using a diagram:



Interference happens because there is a path difference between the two routes taken by the wave. By Pythagoras' Theorem, the path difference is

$$p = 2\sqrt{H^2 + \frac{D^2}{4}} - D$$

A full cycle of fluctuation will occur each time this path difference varies over a whole wavelength of the radiation. In other words, if we have observed $\mathrm{d}n$ fluctuations, the change in path difference will be $\mathrm{d}p = \lambda \mathrm{d}n$. Differentiating both sides with respect to p, we get

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \lambda \frac{\mathrm{d}n}{\mathrm{d}t}$$

Note that $\mathrm{d}n/\mathrm{d}t$ is simply $f_{\text{fluctuations}}$, the frequency of the fluctuations (=the number of fluctuations per second), which in our case is $N/60$. So

$$N = \frac{60}{\lambda}\frac{\mathrm{d}p}{\mathrm{d}t}$$

Furthermore, note that the wave we're considering travels at the speed of light, so $\lambda = c/\nu$, and

$$N = \frac{60\nu}{c}\frac{\mathrm{d}p}{\mathrm{d}t}$$

Differentiating the expression for p above, we get

$$N = \frac{60\nu}{c} 2H \frac{\mathrm{d}H}{\mathrm{d}t}\left(H^2 + \frac{D^2}{4}\right)^{-1/2}$$

But of course, $\mathrm{d}H/\mathrm{d}t = V$, the speed at which the ionosphere is moving. Therefore

$$N = \frac{60\nu}{c} 2HV \left(H^2 + \frac{D^2}{4} \right)^{-1/2}$$

Re-arranging, we get

$$\boxed{V = \frac{Nc}{120\nu H}\sqrt{H^2 + \frac{D^2}{4}}}$$

(f) This is a (very) sneaky question, and to solve it correctly, you need to notice that the tension in the string is not uniform; bits higher up the string will be supporting a greater mass and therefore be under greater tension. The wave speed will therefore vary at different points along the rope.

Our strategy will be to consider a point that is a distance X from the **bottom** of the rope and consider the time t_{pulse} it takes for the pulse to get there, and the time t_{body} it takes for the body to get there. Let's start with the easy one:

- **The body** starts at $x = \ell$ and falls through a distance $\ell - X$. Its original velocity is 0 and its acceleration is g, and so using $x = \frac{1}{2}at^2$, we obtain

$$t_{\text{body}}(x) = \sqrt{\frac{2(\ell - X)}{g}}$$

- For **the pulse**, consider that a point a distance X above the bottom of the string supports a mass of string $\frac{M}{\ell}X$ that is lying below it. The force exerted by this mass of string is $\frac{M}{\ell}Xg$, and to balance that weight out, the tension in the string at that point needs to also be $\frac{M}{\ell}Xg$. Thus, the velocity of waves on the string at that point is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v = -\sqrt{T / \rho} = -\sqrt{\frac{Mgx / \ell}{M / \ell}} = -\sqrt{gx}$$

(Negative because the waves are going downwards and we're measuring our distances upwards). And therefore

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{1}{\sqrt{gx}}$$

We can integrate both sides with respect to x between two points: (a) the point at which $t = 0$ and $x = \ell$ [start of pulse] and (b) the point at which $t = t_{\text{pulse}}$ and $x = X$. We get:

$$\int_0^{t_{\text{pulse}}} \mathrm{d}t = -\int_\ell^X \frac{1}{\sqrt{gx}}\,\mathrm{d}x$$

$$t_{\text{pulse}}(x) = 2\sqrt{\ell / g} - 2\sqrt{x / g}$$

To find where they meet, we simply set

$$t_{\text{body}}(x) = t_{\text{pulse}}(x)$$

$$\sqrt{\frac{2(\ell - X)}{g}} = 2\sqrt{\ell / g} - 2\sqrt{x / g}$$

$$\sqrt{2(\ell - X)} = 2\sqrt{\ell} - 2\sqrt{x}$$

Squaring

$$2\left(\ell - X\right) = 4\ell + 4x - 8\sqrt{\ell x}$$

$$\left(\ell - X\right) = 2\ell + 2x - 4\sqrt{\ell x}$$

$$4\sqrt{\ell x} = \ell + 3x$$

Squaring again

$$16\ell x = \ell^2 + 9x^2 + 6\ell x$$

$$9x^2 - 10\ell x + \ell^2 = 0$$

Solving the quadratic, we then get

$$x = \ell \qquad\qquad x = \frac{\ell}{9}$$

$x = \ell$ represents the start of the pulse, when both begin from the same place. The second solution represents the fact that the pulse and body will meet again at a distance $\ell / 9$ from the bottom of the string.

(g) The kinetic and potential energies per unit length in a wave on a string are (see later questions for proofs)

$$\text{KE} = \frac{1}{4}\rho y_0^2\omega^2 \qquad\qquad \text{PE} = \frac{1}{4}\rho y_0^2\omega^2$$

The total energy per unit length is therefore

$$E = \frac{1}{2}\rho y_0^2\omega^2$$

Ever second, the wave travels a distance c. So To "maintain a wave travelling on a string", we must increase the length of string which is moving at a rate c per second. Thus, the power needed is

$$P = \frac{1}{2}c\rho y_0^2\omega^2$$

All that's left to do is to express this in terms of the quantities in the question: y_0, λ, ρ and T. We know that $c = \sqrt{T / \rho}$ and $\omega = ck$ and $k = 2\pi / \lambda$, so

$$P = 2\pi^2 \frac{y_0^2}{\lambda^2}\sqrt{\frac{T^3}{\rho}}$$

(h) If u is the distance of the object from the lens and v is the distance of the image, we know that

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

we also know that the distance between the object and image is kept fixed. Let's call that distance a

$$u + v = a$$

Let's use the second equation to eliminate v from the first

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{a-u}$$

And now solve for u

$$\frac{1}{f} = \frac{a}{u(a-u)}$$

$$fa = ua - u^2$$

$$u^2 - au + fa = 0$$

Solving

$$u = \frac{a \pm \sqrt{a^2 - 4fa}}{2}$$

$$u_1 = \frac{a + \sqrt{a^2 - 4fa}}{2} \qquad u_2 = \frac{a - \sqrt{a^2 - 4fa}}{2}$$

So we see that there are indeed two possible positions for the lens that will lead to the same distance between image and object.

Now, let the size of the object be H and the size of the two images be h_1 and h_2. By similar triangles in each case, we know that

$$\frac{h_1}{v_1} = \frac{H}{u_1} \qquad \frac{h_2}{v_2} = \frac{H}{u_2}$$

Rearranging

$$h_1 = \frac{v_1}{u_1}H \qquad h_2 = \frac{v_2}{u_2}H$$

Now, it's tempting to use $u + v = a$ on the right-hand-side and then simply feed in our expressions for u_1 and u_2 above, but that'd be tedious, since we would have to take the *reciprocal* of these expressions. Instead, we want to try and get rid of the $1/u$ bits. We do that by using

$$\frac{1}{u} = \frac{1}{f} - \frac{1}{v}$$

The expressions above then become

$$h_1 = \left(\frac{v_1}{f} - 1\right)H \qquad h_2 = \left(\frac{v_2}{f} - 1\right)H$$

And we can finally write

$$h_1 = \left(\frac{a - u_1}{f} - 1\right)H \qquad h_2 = \left(\frac{a - u_2}{f} - 1\right)H$$

Feeding in our expressions for u_1 and u_2 above, we get

$$h_1 = \left(a - 2f - \sqrt{a^2 - 4fa}\right)\frac{H}{2f}$$

$$h_2 = \left(a - 2f + \sqrt{a^2 - 4fa}\right)\frac{H}{2f}$$

Finally, we can calculate

$$h_1 h_2 = H^2$$

As required.

(i) Most of the terms are defined in the small section at the start of this document. Here are the extra ones:

- **Amplitude**: describes the "strength" of the wave. Formally, it's a measure of the distance from the largest point of the wave to the average point of the wave.
- **Wavevector**: for a wave travelling in more than one dimension, the wavevector encompasses both the wavenumber and the direction of the wave. For example, consider a wave of wavelength λ travelling at 15° to the $\boldsymbol{i}$ direction. The wavevector for that wave is

$$\boldsymbol{k} = \frac{2\pi}{\lambda}\cos(15)\boldsymbol{i} + \frac{2\pi}{\lambda}\sin(15)\boldsymbol{j}$$

The form of the wave is then

$$\sin\left(\omega t - \boldsymbol{k}\cdot\boldsymbol{x}\right)$$

Question 2 (Tripos 2005)

The transverse displacement y for waves traveling in the x direction on a string under tension T is given by

$$y = y_0 \cos\left(\omega t - kx\right)$$

where ω and k are both positive. Explain why this describes the velocity of a wave traveling in the positive x direction, and show that the velocity of the wave, c, is equal to ω / k.

Show that the instantaneous rate of work done by the string to the left of $x = 0$ on the string to the right is given by

$$W = -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = Ty_0^2 k\omega \sin^2(\omega t)$$

show that this result is consistent with the expression $\frac{1}{2}y_0^2 k^2 T$ for the mean energy per unit length in the wave.

Solution:

To see why this wave travels in a positive x direction, imagine that a point x on the spring happens to be at a height y at a give time t. Now consider a time $t + \delta t$, and ask what point X on the string will be at a height y at that time. The point X satisfies

$$\omega(t + \delta t) + kX = \omega t + kx$$
$$X = x + \frac{\omega \delta t}{k}$$

Since ω, δt and k are all positive, we clearly have $X > x$. Thus, the point has moved in the positive x direction, and so has the wave.

To work out the velocity of the wave, simply consider that from the equation above, the point has moved a distance $\omega\delta t / k$ in a time δt. Thus

$$\text{Speed} = \text{Distance/Time} = \omega / k$$

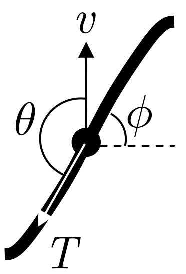
Alternatively, you could have used $\lambda = 2\pi / k$, $\nu = \omega / 2\pi$ and $c = \nu\lambda$ to get the same result.

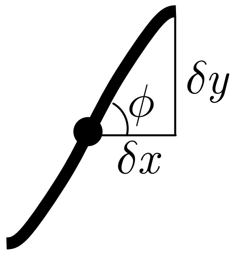
Alternatively, you could have fed the expression directly into the wave equation

$$y'' = \frac{1}{c^2}\ddot{y}$$
$$-k^2 y_0 \cos\left(\omega t - kx\right) = -\frac{1}{c^2}\omega^2 y_0 \cos\left(\omega t - kx\right)$$
$$c = \omega / k$$

as above.

Consider the point at $x = 0$. It is moving with a velocity v, and a force of magnitude T is acting on it:





Now, $v = \partial y / \partial t$, and as you'll remember from the Michaelmas mechanics course, the instantaneous rate of work done by such a force is

$$\begin{aligned} W = \boldsymbol{v}\cdot\boldsymbol{F} &= \frac{\partial y}{\partial t} T\cos(\theta) \\ &= -\frac{\partial y}{\partial t} T\sin(\theta - \tfrac{\pi}{2}) \\ &= -\frac{\partial y}{\partial t} T\sin(\phi) \\ &\approx -\frac{\partial y}{\partial t} T\tan(\phi) \\ &= -\frac{\partial y}{\partial t} T\frac{\delta y}{\delta x} \\ &= -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \end{aligned}$$

Feeding in our expression for y, we directly get

$$W = Ty_0^2 k\omega \sin^2(\omega t - kx)$$

but of course, we need to evaluate this at $x = 0$, because that's where we're applying the force, so

$$W = Ty_0^2 k\omega \sin^2(\omega t)$$

The last part of the question is tricky. Consider the following argument step-by-step:

- First, note that the average of $\sin^2(\omega t) = \frac{1}{2}$. To see why, consider that to find the "average" of that function, we need to integrate it over its period, and the period of that function is $\frac{2\pi}{\omega} \div 2 = \pi / \omega$. So

$$\begin{aligned} \text{Average} &= \frac{1}{\pi / \omega}\int_0^{\pi/\omega} \sin^2(\omega t)\ \mathrm{d}t = \frac{1}{\pi / \omega}\frac{1}{2}\int_0^{\pi/\omega} 1 - \cos(2\omega t)\ \mathrm{d}t \\ &= \frac{\omega}{2\pi}\left[t - \frac{1}{2\omega}\sin(2\omega t)\right]_0^{\pi/\omega} = \frac{\omega}{2\pi}\left(\frac{\pi}{\omega} - 0 - 0 - 0\right) \\ &= \frac{1}{2} \end{aligned}$$

  This means that the average energy transmitted to the string to the right of $x = 0$ is given by

$$W = \frac{1}{2}Ty_0^2 k\omega$$

  Keep this fact at the back of your mind for a minute…

- Now, imagine the wave is just starting up, and traveling to the right. Imagine it arrives to the point $x = 0$ just as $t = 0$. At that moment, there is 0 energy to the right of $x = 0$.
- We know the wave travels at a speed $c = \omega / k$, and this means that it'll take a time $1 / c = k / \omega$ for it to travel a unit distance past $x = 0$.
- Using the expression above for average energy transmitted per unit time, we therefore have

$$E_{\text{unit length}} = \frac{1}{2}Ty_0^2 k^2$$

Precisely as expected.

Another way to look at it is to realize that to maintain the wave traveling past $x = 0$, we must increase the length of moving string at a rate c (length per unit time). Thus, the total power we need to pass to the string past x_0 is c times the energy per unit length. Thus, $W = cE_{\text{unit length}}$.

Question 3 (Tripos 2007)

Show that the velocity of transverse waves on a string of density ρ per unit length and under tension T is

$$v = \sqrt{T / \rho}$$

for a continuous traveling wave of the form $y = a\sin(\omega t - kx)$ along a string stretched along the x axis, show that the mean kinetic energy per unit length is

$$KE = \tfrac{1}{4}\rho\omega^2 a^2$$

A string is stretched along the x-axis and is at rest. A pulse of sinusoidal shape is then generated on the string by moving one end of it (at $x = 0$) through the displacement

$$y = a\sin\omega t \qquad \text{for } 0 \le t \le 2\pi / \omega$$

show that the work done in generating the pulse is

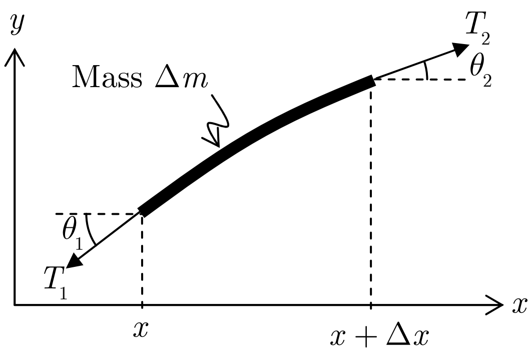
$$W = \pi v\rho\omega a^2$$

Hence show that the mean potential energy per unit length for a wave of the form $y = a\sin(\omega t - kx)$ is

$$PE = \tfrac{1}{4}\rho\omega^2 a^2$$

Solution:

Consider a small segment of the string, in all generality



let's apply Newton's Second Law to the vertical motion of the segment:

$$T_2 \sin\theta_2 - T_1 \sin\theta_1 = \Delta m \frac{\partial^2 y}{\partial t^2}$$

We now make a few assumptions, all of which rely on the fact that we're only considering *very small* waves along the string:

- The tension is constant throughout the string, so $T_1 = T_2 = T$
- The mass of the segment is given by $\Delta m = \rho \Delta x$

- The small angle approximation applies, so that

$$\sin\theta = \tan\theta = \left(\frac{\partial y}{\partial x}\right)$$

for both our angles.

Our equation then becomes:

$$\begin{aligned} T\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - T\left(\frac{\partial y}{\partial x}\right)_x &= \rho \Delta x \frac{\partial^2 y}{\partial t^2} \\ T\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\} &= \rho \Delta x \frac{\partial^2 y}{\partial t^2} \\ T\Delta\left(\frac{\partial y}{\partial x}\right) &= \rho \Delta x \frac{\partial^2 y}{\partial t^2} \\ \frac{\Delta\left(\frac{\partial y}{\partial x}\right)}{\Delta x} &= \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \end{aligned}$$

As $\Delta x \to 0$, this becomes

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

This is a wave equation, with

$$\begin{aligned} \frac{1}{c^2} &= \frac{\rho}{T} \\ \Rightarrow c &= \sqrt{T / \rho} \end{aligned}$$

To find the kinetic energy, consider once again our segment. The kinetic energy stored therein is

$$\begin{aligned} \text{KE} &= \frac{1}{2}\Delta m \left(\frac{\partial y}{\partial t}\right)^2 \\ &= \frac{1}{2}\rho \Delta x \left(\frac{\partial y}{\partial t}\right)^2 \end{aligned}$$

Assuming we have a wave of the form $y = a\sin(\omega t - kx)$, this becomes

$$\text{KE} = \frac{1}{2}\rho \Delta x a^2 \omega^2 \cos^2(\omega t - kx)$$

To find the *average* energy in this segment, we remember that the average of $\cos^2$ is ½ (see previous question), so

$$\text{KE}_{\text{average}} = \frac{1}{4}\rho \Delta x a^2 \omega^2$$

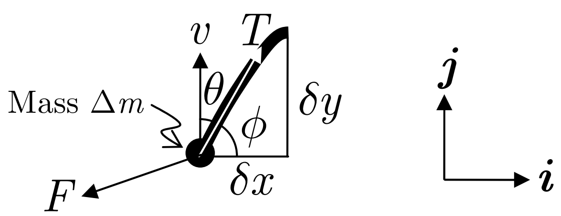
And finally, to find the average energy per unit length, we divide by the length of this segment (namely, Δx):

$$\text{KE}_{\text{u. length}} = \frac{1}{4}\rho a^2\omega^2$$

For the next part, we first need to ask what kind of pulse is propagating along the wire. We now we're forcing the *end* with $y = a\sin\omega t$, but how does that pulse then propagate along the wave? To answer the question, consider the following steps

- We know that at $x = 0$
  $$y(0,t) = a\sin\omega t$$
  because that's the way we're forcing it.
- We further know that the wave travels at a speed c. It will therefore take the wave a time x/c to travel a distance x along the wire.
- Therefore, if we look at a point x along the wire at a time t, the height of the wire there will be equal to the height of the wire at $x = 0$ when the time was $t - (x/c)$
- Therefore
  $$y(x,t) = y(0,t - \tfrac{x}{c}) = a\sin\left(\omega\left[t - \tfrac{x}{c}\right]\right)$$
- But $c = \omega / k$, so
  $$y(x,t) = a\sin\left(\omega t - kx\right)$$

Now we need to find the work being done. Consider the end of the string that we're moving up and down. $\boldsymbol{F}$ is the force we're applying, of unknown magnitude and direction:



By Newton's Second Law in Two Dimensions, we have

$$\boldsymbol{F} = \boldsymbol{a}\Delta m$$

$$\boldsymbol{T} + \boldsymbol{F} = \Delta m\frac{\partial^2 y}{\partial t^2}\boldsymbol{j}$$

$$\boldsymbol{F} = \Delta m\frac{\partial^2 y}{\partial t^2}\boldsymbol{j} - \boldsymbol{T}$$

And as $\Delta m \to 0$,

$$\boldsymbol{F} = -\boldsymbol{T}$$

This implies that the force we're applying is equal in magnitude to the tension but opposite in direction[1]. We can then easily find the power exerted by that force; the calculation is identical to what we did in the last question, and the result is therefore

$$P = -T\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}$$

Feeding our expression $y(x,t) = a\sin\left(\omega t - kx\right)$ into this, we obtain

$$P = Ta^2\omega k\cos^2(\omega t - kx)$$

But of course, we need to evaluate this at $x = 0$, because that's where we're applying the force, so

$$P = Ta^2\omega k\cos^2(\omega t)$$

The pulse is applied for a time $0 \le t \le 2\pi / \omega$, and so the total work done is

$$\begin{aligned} W &= \int_0^{2\pi/\omega} P \ \mathrm{d}t \\ &= Ta^2\omega k\int_0^{2\pi/\omega} \cos^2(\omega t) \ \mathrm{d}t \\ &= \frac{1}{2}Ta^2\omega k\int_0^{2\pi/\omega} 1 + \cos(2\omega t) \ \mathrm{d}t \\ &= \frac{1}{2}Ta^2\omega k\left[t + \frac{1}{2\omega}\sin(2\omega t)\right]_0^{2\pi/\omega} \\ &= T\pi a^2 k \end{aligned}$$

But

$$\begin{aligned} k &= \omega / v \\ &= \omega\sqrt{\frac{\rho}{T}} \\ &= \omega v\rho / T \end{aligned}$$

and so

$$W = \pi v\rho\omega a^2$$

---

[1] Wait a second! How on earth can the applied force be equal to the tension; surely, that would mean the string isn't accelerating at all! Nope, simply because the minute point at the end of the string has 0 mass. This means that even if it's accelerating, it *must* have 0 resultant force on it, because $F = ma = 0 \times a = 0$. In fact, if there *was* any force acting on it, its acceleration would be infinite! As soon as you move away from the minute point at the end of the string, however, two things happen (1) the mass increases from 0 (2) the direction of the tension changes, which means that now you *do* have a resultant force in a given direction. So all's good!

The pulse generated lasts a time $\frac{2\pi}{\omega}$ and therefore travels a distance $\frac{2\pi v}{\omega}$. Thus, the total energy *per unit length* is

$$\frac{W}{2\pi v \,/\, \omega} = \frac{1}{2}\rho\omega^2 a^2$$

As expected.

Question 4

The second and third harmonics of a pipe are 525 Hz and 875 Hz. Find the fundamental frequency of the pipe.

Solution:

There are three factors that affect the harmonics in a pipe:

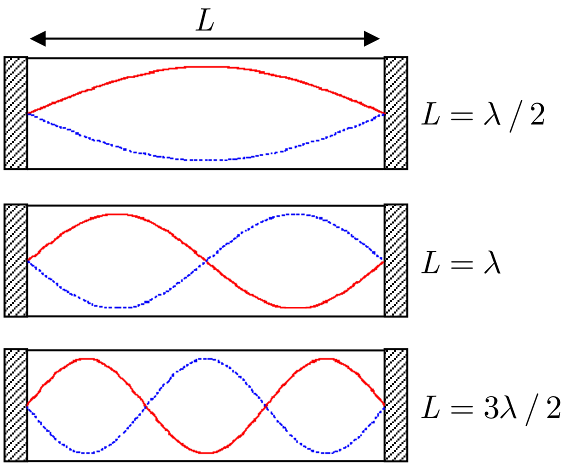
- The ratio of the speed of sound in the pipe and the length of the pipe, v/L.
- Whether the pipe is open or closed, at each end.

That's two factors, and the question supplies two harmonic frequencies, which will lead to two equations. So hopefully, we'll be able to find all the quantities above and determine the fundamental.

Let's consider each of the three possible types of pipes:

Closed Pipe

The first three harmonics in a closed pipe look like this:



Assuming our wave has the form

$$\psi(x,t) = \left(A\cos kx + B\sin kx\right)\cos\omega t$$

the boundary conditions for a closed pipe are

$$\psi(0,t) = \left(A\right)\cos\omega t = 0 \Rightarrow A = 0$$

and

$$\psi(L,t) = B\sin kL\cos\omega t = 0 \Rightarrow kL = n\pi$$

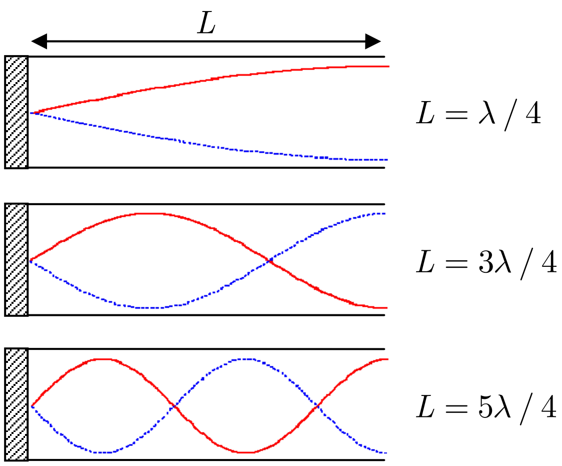
Now, we know that $k = 2\pi / \lambda$ and that $v = f\lambda$, so $k = 2\pi f / v$ and so the above condition becomes

$$\boxed{f_n = \frac{nv}{2L}}$$

(Much more informally, all we're saying is that it's clear from the diagram that L needs to be an integer multiple of $\lambda / 2$, so $L = n\lambda / 2$, and since $v = f\lambda$, we get the expression above).

Pipe Open at One End

The first three harmonics in a pipe open at one end look like this:



Assuming our wave has the form

$$\psi(x,t) = \left(A\cos kx + B\sin kx\right)\cos\omega t$$

the boundary conditions for this pipe are

$$\psi(0,t) = \left(A\right)\cos\omega t = 0 \Rightarrow A = 0$$

and

$$\frac{\partial\psi}{\partial x}(L,t) = Bk\cos kL\cos\omega t = 0 \Rightarrow kL = \frac{(2n-1)\pi}{2}$$

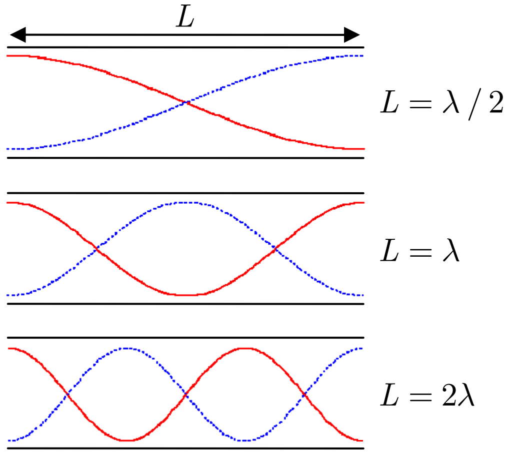
Now, we know that $k = 2\pi / \lambda$ and that $v = f\lambda$, so $k = 2\pi f / v$ and so the above condition becomes

$$\boxed{f_n = \frac{(2n-1)v}{4L}}$$

(Much more informally, all we're saying is that it's clear from the diagram that L needs to be an odd integer multiple of $\lambda / 4$, so $L = (2n+1)\lambda / 4$, and since $v = f\lambda$, we get the expression above).

Pipe Open at Both Ends

The first three harmonics in a pipe open at both ends look like this:



Assuming our wave has the form

$$\psi(x,t) = \left(A\cos kx + B\sin kx\right)\cos\omega t$$

the boundary conditions for a closed pipe are

$$\frac{\partial\psi}{\partial x}\psi(0,t) = \left(Bk\right)\cos\omega t = 0 \Rightarrow B = 0$$

and

$$\frac{\partial\psi}{\partial x}(L,t) = -Ak\sin kL\cos\omega t = 0 \Rightarrow kL = n\pi$$

Now, we know that $k = 2\pi / \lambda$ and that $v = f\lambda$, so $k = 2\pi f / v$ and so the above condition becomes

$$\boxed{f_n = \frac{nv}{2L}}$$

(Much more informally, all we're saying is that it's clear from the diagram that L needs to be an integer multiple of $\lambda / 2$, so $L = n\lambda / 2$, and since $v = f\lambda$, we get the expression above).

Back to the Question!

OK, so we've seen that the harmonics of a pipe that is open/closed at both ends are the same, and different from those of a pipe open at one end and closed at the other.

Let's go back to the question, our strategy will be:

- Assume a "pipe type"
- Use the 2nd harmonic to work out v/L
- Check if the 3rd harmonic is consistent.

Let's go for it!

- **Assume the pipe is closed/open at both ends**
  - The 2nd harmonic would then be
    $$f_2 = v / L = 525$$
  - The 3rd harmonic should then be
    $$f_3 = \tfrac{3}{2}\tfrac{v}{L} = \tfrac{3}{2}\times 525 = 787.5 \text{ Hz}$$
  - This is not the 3rd harmonic we're given in the question, so clearly that was a wrong assumption.
- **Assume the pipe is closed at one end and open at the other**
  - The 2nd harmonic would then be
    $$f_2 = \tfrac{3}{4}\tfrac{v}{L} = 525 \Rightarrow 700 \text{ Hz}$$
  - The 3rd harmonic should then be
    $$f_3 = \tfrac{5}{4}\tfrac{v}{L} = \tfrac{5}{4}\times 700 = 875 \text{ Hz}$$
  - Hurray!

The fundamental frequency is therefore

$$f_1 = \frac{1}{4}\frac{v}{L} = 175 \text{ Hz}$$

Question 5 (Tripos 2009)

In quantum mechanics, what is the physical significance of the wavefunction?

The wavefunction must satisfy the three-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

In the one-dimensional case, for a particle having constant energy E and moving in the x-direction in a region of constant uniform potential V, show that the stationary state

$$\Psi(x,t) = Ae^{i(kx-\omega t)}$$

is a solution to Schrödinger's Equation, and derive an expression for k.

A particle moving in the positive x direction with energy E is incident on a potential step, where $V = V_1$ for $x < 0$, $V = V_0$ for $x \geq 0$, and $E > V_1 > V_0$. What constraints must the wavefunction satisfy at the step?

In the regions before and after the step, the wavefunction has the general form

$$\Psi(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$$

with different parameters A, B and k on each side of the step. What is the value of B in the region $x \geq 0$?

Derive an expression for the probability of reflection $R = \left|B / A\right|^2$ and show that, as the step height tends to 0, the solution tends to the classical result.

Why is the non-zero reflection coefficient difficult to understand classically?

Solution:

The function $f(\boldsymbol{x}) = \psi(\boldsymbol{x})\psi^*(\boldsymbol{x})$, where $\psi(\boldsymbol{x})$ is the wavefunction, is the probability density function of the position of the particle described by the wavefunction. In other words, the probability that the particle will be found in a volume V is given by

$$P = \int_V \psi(\boldsymbol{x})\psi^*(\boldsymbol{x})\ \mathrm{d}^3\boldsymbol{x}$$

Let's calculate some derivatives

$$\frac{\partial \Psi}{\partial x} = Aike^{i(kx-\omega t)} \Rightarrow \frac{\partial^2 \Psi}{\partial x^2} = -Ak^2 e^{i(kx-\omega t)}$$
$$\frac{\partial \Psi}{\partial t} = -Ai\omega e^{i(kx-\omega t)}$$

Feeding these results into the Schrodinger Equation

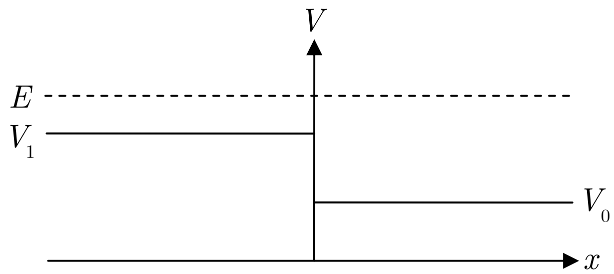
$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}$$
$$\frac{\hbar^2}{2m}Ak^2 e^{i(kx-\omega t)} + AVe^{i(kx-\omega t)} = -i\hbar Ai\omega e^{i(kx-\omega t)}$$
$$\frac{\hbar^2}{2m}k^2 + V = \hbar\omega$$
$$\boxed{k = \sqrt{\frac{2m}{\hbar^2}\left(\hbar\omega - V\right)}}$$

Given that $E = \hbar\omega$, we can re-write this as

$$\boxed{k = \sqrt{\frac{2m}{\hbar^2}\left(E - V\right)}}$$

This makes sense – $(E - V)$ is the total kinetic energy of the particle, which makes its velocity $v = \sqrt{2(E-V)/m}$, which makes its momentum $p = mv = \sqrt{2m(E-V)}$. We then obtain that $k = p/\hbar$, which we know is true.

The step looks like this:



At the step:

- The wavefunction must be continuous
- The derivative of the wavefunction must be continuous

Clearly, $B = 0$ for $x \geq 0$, because there is no "backwards going wave" for $x \geq 0$, because it couldn't have come from anywhere. We therefore have

$$\Psi_{x\leq 0}(x,t) = A_{x\leq 0}e^{i(k_{x\leq 0}x-\omega t)} + B_{x\leq 0}e^{-i(k_{x\leq 0}x+\omega t)}$$
$$\Psi_{x\geq 0}(x,t) = A_{x\geq 0}e^{i(k_{x\geq 0}x-\omega t)}$$

From the previous part of the question, we know that

$$k_{x\le 0} = \sqrt{\frac{2m}{\hbar^2}\left(E - V_1\right)} \qquad\qquad k_{x\ge 0} = \sqrt{\frac{2m}{\hbar^2}\left(E - V_0\right)}$$

Now, continuity of the wavefunction at $x = 0$ implies that

$$\Psi_{x\le 0}(0,t) = \Psi_{x\ge 0}(0,t)$$
$$A_{x\le 0}e^{-i\omega t} + B_{x\le 0}e^{-i\omega t} = A_{x\ge 0}e^{-i\omega t}$$
$$A_{x\le 0} + B_{x\le 0} = A_{x\ge 0}$$

Continuity of the derivative of the wavefunction at $x = 0$ implies that

$$\Psi'_{x\le 0}(0,t) = \Psi'_{x\ge 0}(0,t)$$
$$iA_{x\le 0}k_{x\le 0}e^{-i\omega t} - iB_{x\le 0}k_{x\le 0}e^{-i\omega t} = iA_{x\ge 0}k_{x\ge 0}e^{-i\omega t}$$
$$\left(A_{x\le 0} - B_{x\le 0}\right)k_{x\le 0} = A_{x\ge 0}k_{x\ge 0}$$

We can use the first equation to eliminate $A_{x\ge 0}$:

$$\left(A_{x\le 0} - B_{x\le 0}\right)k_{x\le 0} = \left(A_{x\le 0} + B_{x\le 0}\right)k_{x\ge 0}$$

Now, let's keep in mind that we want to find

$$R = \left|\frac{B_{x\le 0}}{A_{x\le 0}}\right|^2$$

Let's divide the equation above by $A_{x\le 0}$ throughout

$$\left(1 - \frac{B_{x\le 0}}{A_{x\le 0}}\right)k_{x\le 0} = \left(1 + \frac{B_{x\le 0}}{A_{x\le 0}}\right)k_{x\ge 0}$$

Re-arranging

$$k_{x\le 0} - k_{x\ge 0} = \frac{B_{x\le 0}}{A_{x\le 0}}\left(k_{x\ge 0} + k_{x\le 0}\right)$$
$$\frac{B_{x\le 0}}{A_{x\le 0}} = \frac{k_{x\le 0} - k_{x\ge 0}}{k_{x\le 0} + k_{x\ge 0}}$$

This gives

$$R = \left(\frac{k_{x\le 0} - k_{x\ge 0}}{k_{x\le 0} + k_{x\ge 0}}\right)^2$$

If we particularly wanted to, we could feed in specific expressions for the k, and obtain

$$R = \left|\frac{\sqrt{E-V_1}-\sqrt{E-V_0}}{\sqrt{E-V_1}+\sqrt{E-V_0}}\right|^2$$

$$R = \frac{2E-V_1-V_0-2\sqrt{\left(E-V_1\right)\left(E-V_0\right)}}{2E-V_1-V_0+2\sqrt{\left(E-V_1\right)\left(E-V_0\right)}}$$

$$\boxed{R = 1-\frac{4\sqrt{\left(E-V_1\right)\left(E-V_0\right)}}{2E-V_1-V_0-2\sqrt{\left(E-V_1\right)\left(E-V_0\right)}}}$$

As the step height tends to 0, $V_0 \to V_1 = V$, and

$$R = 1-\frac{4\left(E-V\right)}{2E-2V-2\left(E-V\right)} = 1-1 = 0$$

Which is what we would expect classically.

The nonzero reflection coefficient is difficult to understand classically because it implies that particles moving towards a *downwards* step will, somehow, bounce back. It's difficult to think why that might happen, since there's no force to make it happen.

Question 6

The midpoints of two identical vertical slits of width a are positioned at A and C, a distance d apart, where $d > a$. The slits are illuminated by a plane wave of wavelength λ, which is incident normally on the plane of the slits. The intensity of the resulting wave, $I(\theta)$, is observed at a distant point P. The position vector of P from B subtends an angle θ with the perpendicular bisector of the line AC.

Obtain an expression for $I(\theta) / I(0)$. In each of the following cases, plot your answer and explain the result using a phasor diagram

i. $a \to 0$

ii. $d = a$

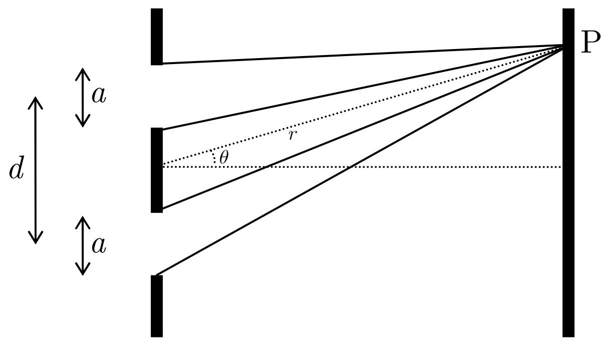
iii. The general case $d > a$

Why would it not make sense to set $d = 0$?

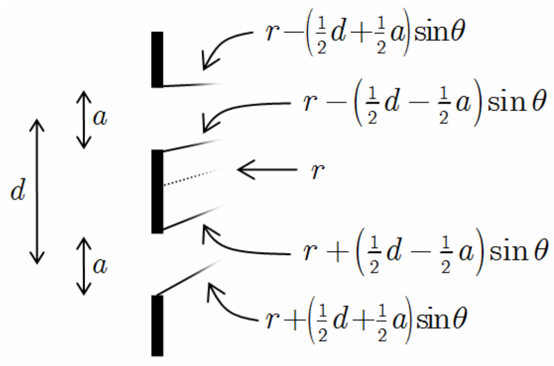
[*Note*: this is not a simple "young's double slit" setup, because each slit has a finite width a]

Solution:

Consider the point P described in the question. The situation is as follows (note: for simplicity, I've only drawn four beams, but of course there'll be beams coming out of *every* part of the slits)



Let the path length from the point in between the slits to P be r, and consider the path lengths of the other beams above:



The above diagram clearly indicates that the path length seems to be varying linearly with the distance of the originating position from the centre of the two slits. In fact, convince yourself that if x is the distance of the originating beam from the centre of the slits (x upwards means positive), then the path length is given by

$$r - x\sin\theta$$

This means that a wave that leaves at a distance x from the centre of the two slits will have the form

$$A\exp\left(i\left[\omega t - k\left\{r - x\sin\theta\right\}\right]\right)$$

[I've expressed the wave in exponential form because it'll make the upcoming integration much easier. You already saw in your SHM course that this is equivalent to expressing it as a sin or cosine].

Now, to find the wavefunction at the screen, $\psi(\theta)$, we want to add *all* such waves coming from both the slits. But because the slits are continuous, we'll need to *integrate* to take into account *all* the beams coming out. In fact

$$\psi(\theta) = \int_{-\left(\frac{1}{2}d+\frac{1}{2}a\right)}^{-\left(\frac{1}{2}d-\frac{1}{2}a\right)} A\exp\left(i\left[\omega t - k\left\{r - x\sin\theta\right\}\right]\right)\,\mathrm{d}x + \int_{\left(\frac{1}{2}d-\frac{1}{2}a\right)}^{\left(\frac{1}{2}d+\frac{1}{2}a\right)} A\exp\left(i\left[\omega t - k\left\{r - x\sin\theta\right\}\right]\right)\,\mathrm{d}x$$

The first term in the above comes from the bottom slit, the second from the top slit. We can re-write this as

$$\psi(\theta) = Ae^{i\omega t}e^{-ikr}\left\{\int_{-\left(\frac{1}{2}d+\frac{1}{2}a\right)}^{-\left(\frac{1}{2}d-\frac{1}{2}a\right)} A\exp\left(ikx\sin\theta\right)\mathrm{d}x + \int_{\left(\frac{1}{2}d-\frac{1}{2}a\right)}^{\left(\frac{1}{2}d+\frac{1}{2}a\right)} A\exp\left(ikx\sin\theta\right)\mathrm{d}x\right\}$$

Note, however, that:

- e^{-ikr} is simply a constant, because r doesn't vary too much as you vary θ, in the small angle approximation.

- We can ignore the variation with respect to time, because what you see on the screen is an *averaging* of the light intensity over time. You don't care if the light intensity fluctuates lots of times every second…

As a result, the two pre-factors just give extra constants, and we obtain:

$$\psi(\theta) = A_0 \left\{ \int_{-\left(\frac{1}{2}d+\frac{1}{2}a\right)}^{-\left(\frac{1}{2}d-\frac{1}{2}a\right)} \exp\left(ikx\sin\theta\right)\,\mathrm{d}x + \int_{\left(\frac{1}{2}d-\frac{1}{2}a\right)}^{\left(\frac{1}{2}d+\frac{1}{2}a\right)} \exp\left(ikx\sin\theta\right)\,\mathrm{d}x \right\}$$

Now it's time to integrate! (It's not as bad as it looks, I promise!)

$$\psi(\theta) = \frac{A_0}{ik\sin\theta}\left\{ \left[\exp\left(ikx\sin\theta\right)\right]_{-\left(\frac{1}{2}d+\frac{1}{2}a\right)}^{-\left(\frac{1}{2}d-\frac{1}{2}a\right)} + \left[\exp\left(ikx\sin\theta\right)\right]_{\left(\frac{1}{2}d-\frac{1}{2}a\right)}^{\left(\frac{1}{2}d+\frac{1}{2}a\right)} \right\}$$

Now, remember that $1/i = -i$, and apply the limits:

$$\psi(\theta) = -\frac{A_0 i}{k\sin\theta}\left\{ \begin{array}{l} \exp\left(-ik\left(\frac{1}{2}d-\frac{1}{2}a\right)\sin\theta\right) - \exp\left(-ik\left(\frac{1}{2}d+\frac{1}{2}a\right)\sin\theta\right) \\ \qquad + \exp\left(ik\left(\frac{1}{2}d+\frac{1}{2}a\right)\sin\theta\right) - \exp\left(ik\left(\frac{1}{2}d-\frac{1}{2}a\right)\sin\theta\right) \end{array} \right\}$$

And now recognize some hyperbolic functions!

$$\psi(\theta) = \frac{2A_0}{k\sin\theta}\left\{ i\sinh\left(ik\left(\tfrac{1}{2}d-\tfrac{1}{2}a\right)\sin\theta\right) - i\sinh\left(ik\left(\tfrac{1}{2}d+\tfrac{1}{2}a\right)\sin\theta\right) \right\}$$

Remember that $i\sinh(ix) = -\sin x$, and get

$$\psi(\theta) = \frac{2A_0}{k\sin\theta}\left\{ \sin\left(k\left(\tfrac{1}{2}d+\tfrac{1}{2}a\right)\sin\theta\right) - \sin\left(k\left(\tfrac{1}{2}d-\tfrac{1}{2}a\right)\sin\theta\right) \right\}$$

Just like we did in the double-slit experiment, use the fact that $\sin(A) - \sin(B) = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$, and get

$$\psi(\theta) = \frac{4A_0}{k\sin\theta}\sin\left(k\frac{a}{2}\sin\theta\right)\cos\left(k\frac{d}{2}\sin\theta\right)$$

We can re-write this in terms of the sinc function:

$$\psi(\theta) = \frac{4\frac{a}{2}A_0}{k\frac{a}{2}\sin\theta}\sin\left(k\frac{a}{2}\sin\theta\right)\cos\left(k\frac{d}{2}\sin\theta\right)$$

$$\psi(\theta) = 2aA_0\,\mathrm{sinc}\left(k\frac{a}{2}\sin\theta\right)\cos\left(k\frac{d}{2}\sin\theta\right)$$

We now remember than the intensity is given by the amplitude squared, so

$$I(\theta) = \left[2aA_0\,\mathrm{sinc}\left(k\frac{a}{2}\sin\theta\right)\cos\left(k\frac{d}{2}\sin\theta\right)\right]^2$$

Finally, note that

$$I(0) = \left[2aA_0\,\mathrm{sinc}\left(k\frac{a}{2}0\right)\cos\left(k\frac{d}{2}0\right)\right]^2 = \left[2aA_0\right]^2$$

And so

$$\boxed{\frac{I(\theta)}{I(0)} = \left\{\operatorname{sinc}\left(k\frac{a}{2}\sin\theta\right)\cos\left(k\frac{d}{2}\sin\theta\right)\right\}^2}$$

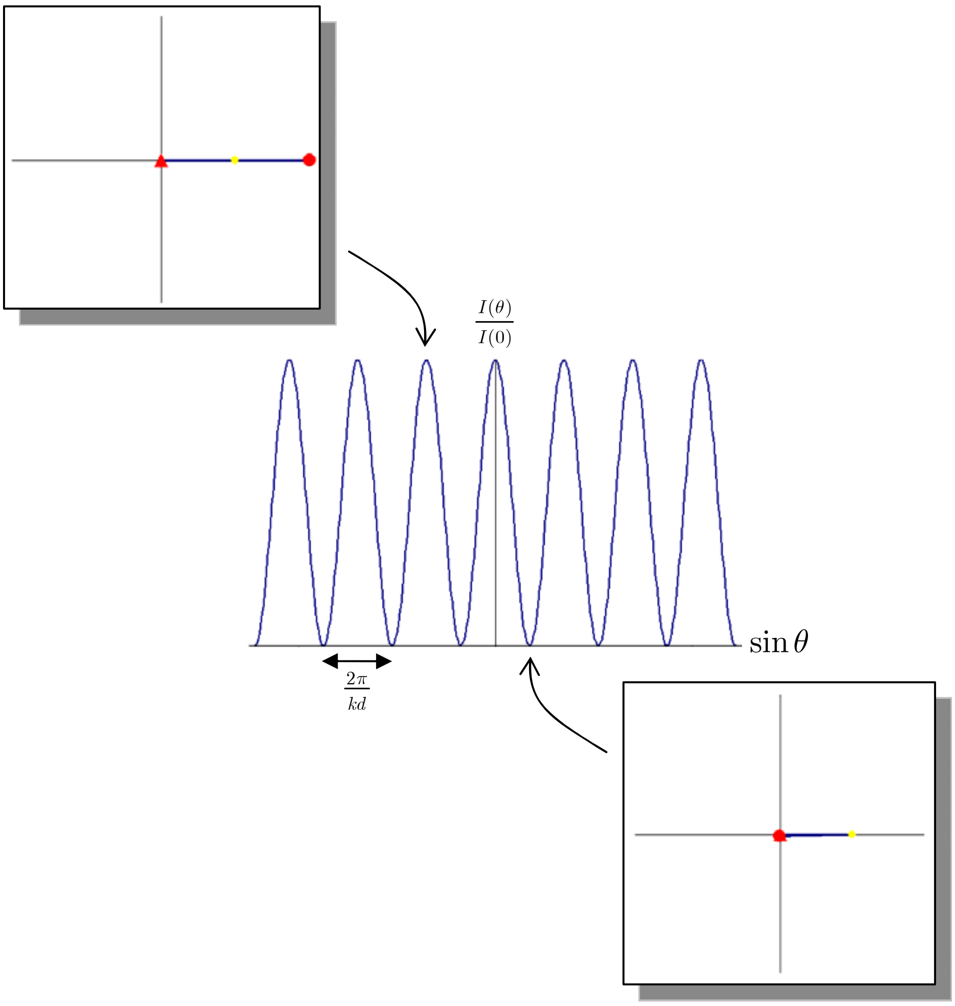
Consider the three cases in the question:

<u>***a* → 0**</u>

In this case, we just have two very thin slits – this is simply Young's double-slit experiment. The formula becomes:

$$\frac{I(\theta)}{I(0)} = \cos^2\left(k\frac{d}{2}\sin\theta\right)$$

The phasor diagram consists of two vectors only; when they're fully aligned, we get maxima and when they're fully anti-aligned, we get minima:



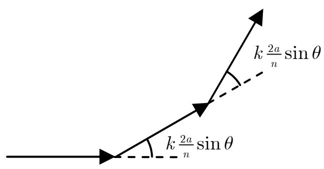
$\underline{d = a}$

In the case $d = a$, there is no gap between the slits, and we just have one large slit of width $2a$. The formula becomes:

$$\begin{aligned}
\frac{I(\theta)}{I(0)} &= \left\{\operatorname{sinc}\left(k\frac{a}{2}\sin\theta\right)\cos\left(k\frac{a}{2}\sin\theta\right)\right\}^2 \\
&= \left\{\frac{1}{k\frac{a}{2}\sin\theta}\sin\left(k\frac{a}{2}\sin\theta\right)\cos\left(k\frac{a}{2}\sin\theta\right)\right\}^2 \\
&= \left\{\frac{1}{k\frac{a}{2}\sin\theta}\frac{1}{2}\sin\left(ka\sin\theta\right)\right\}^2 \\
&= \left\{\operatorname{sinc}\left(ka\sin\theta\right)\right\}^2
\end{aligned}$$

This is *precisely* what we would expect for a single slit of width $2a$!

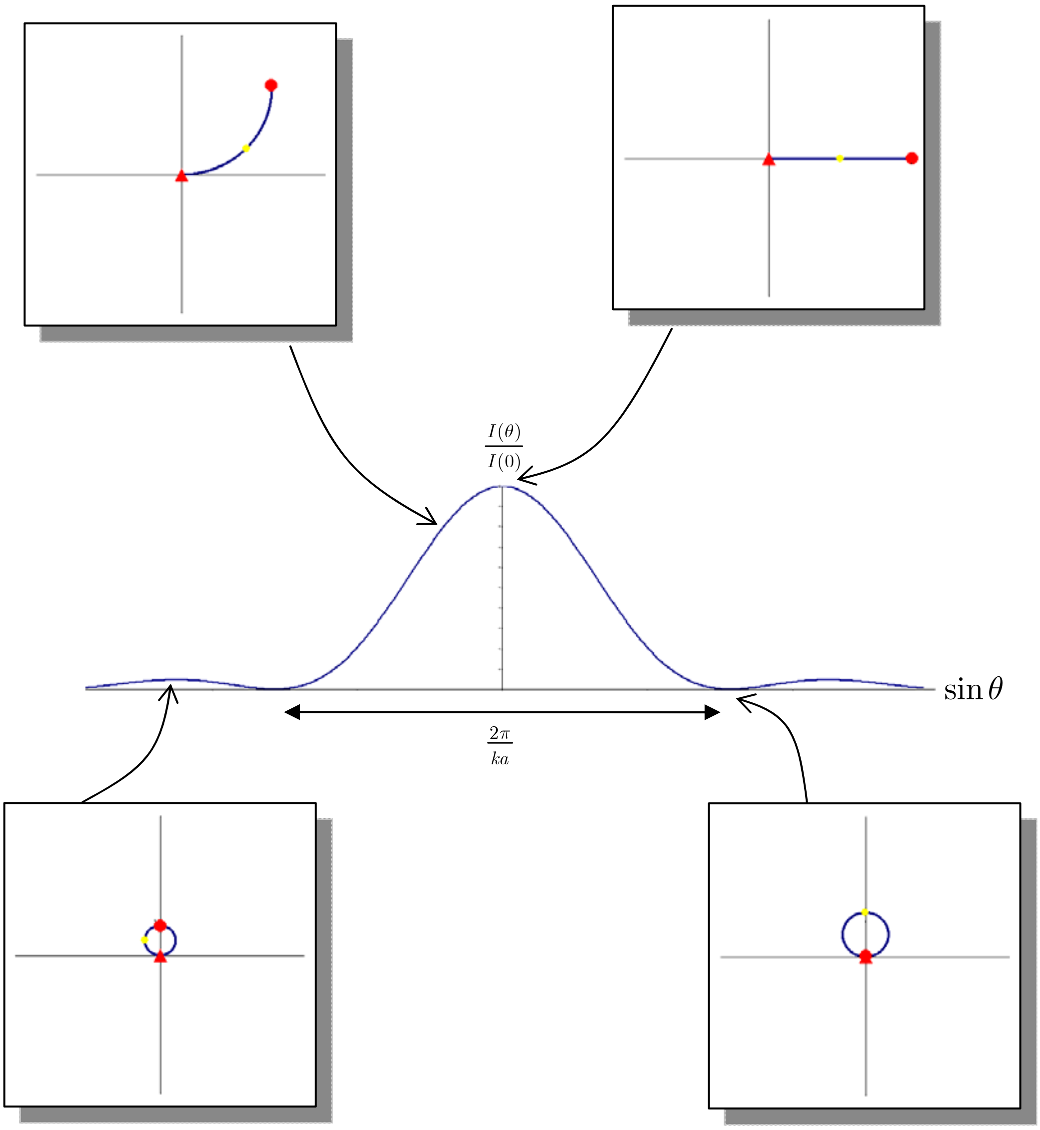
The phasor diagram is harder to understand. Effectively, we split the slit into n little bits, each of size $\frac{2a}{n}$, and we assume that one ray is coming from each of the bits. Each of those rays is then a vector in our phasor diagram, and the phase difference between each bit is, as usual, $k\frac{2a}{n}\sin\theta$. For example, if we choose $n = 3$, we're splitting our large slit into three smaller ones and the phasor diagram would look like this:



As n gets bigger and bigger, more and more vectors are added, and the phase differences get smaller and smaller. Eventually, the phase diagram just looks like the arc of a circle.

- When $\sin\theta = 0$, all we have is a straight line; so we get a maximum.
- As $\sin\theta$ increases, the arc curls further and further upon itself, until it becomes a circle; at that point, the resultant vector (from the tail of the first vector to the tip of the last one) is 0, and we have a minimum.
- As $\sin\theta$ gets even bigger, the circle curls further upon itself and gets smaller and smaller, so we get every decreasing maxima.

This is all illustrated in the diagram below [note: the height of the subsidiary maxima is exaggerated to make them easier to see]:



The general case d > a

In the general case, there's nothing to make our life easier. The formula is

$$\frac{I(\theta)}{I(0)} = \left\{ \operatorname{sinc}\left(k\frac{a}{2}\sin\theta \right) \cos\left(k\frac{d}{2}\sin\theta \right) \right\}^2$$

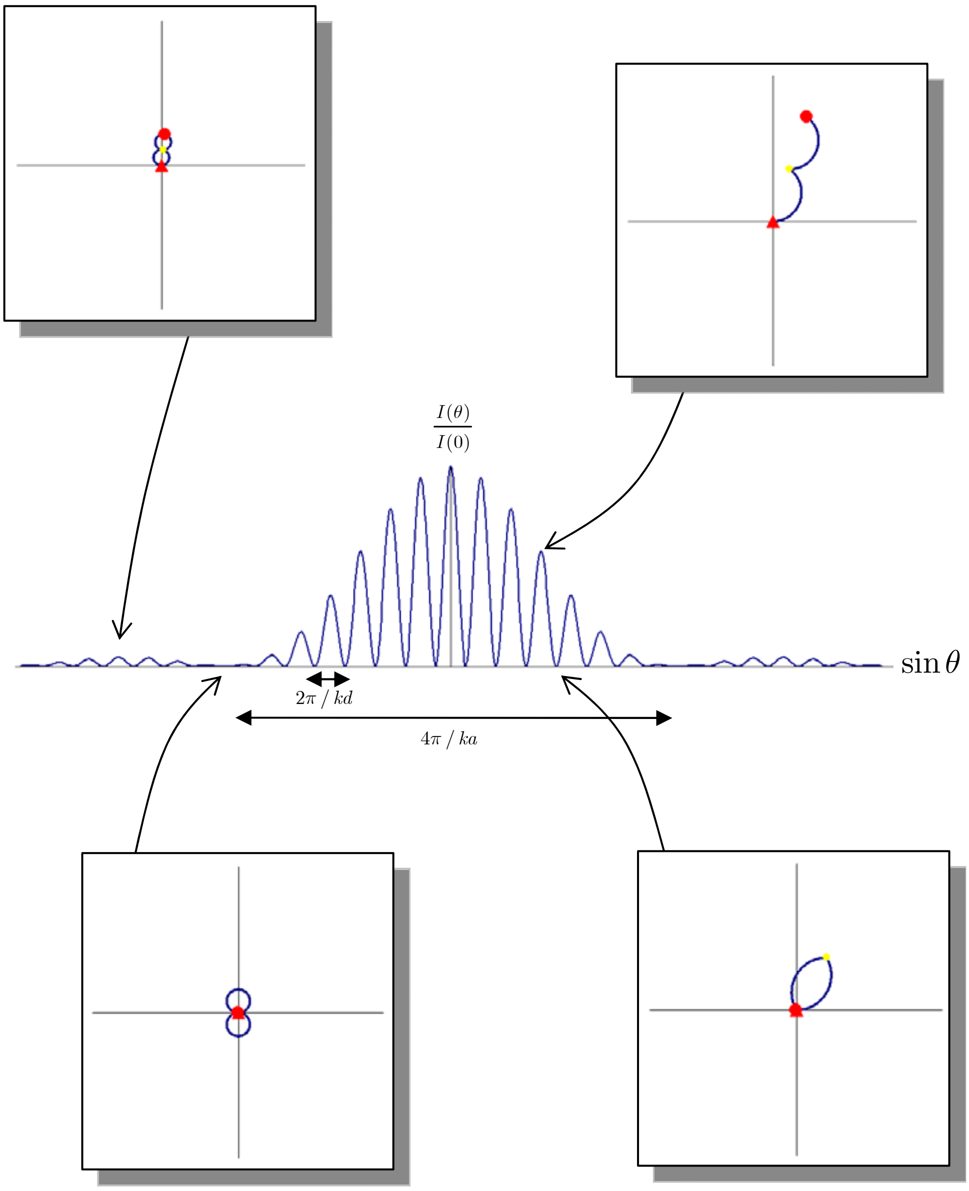
The phasor diagrams will look as follows:

- Each of the two slits will still produce an arc of a circle, because we can still divide each one into lots of tiny segments.

- However, *between* the end of one slit and the beginning of the next, we now have a big gap, which causes a bit phase change. This will appear as a "kink" in our phasor diagram.

So our phasor diagram will look like two arcs of a circle joined to each other but with a kink in the middle. Our minima will then occur for one of two reasons:

- The "kink" will be such that the two ends of the arcs meet (this will be a "Young's double slit style minimum")
- The arcs will wrap upon themselves to make a circle (this will be a "wide slit type minimum")



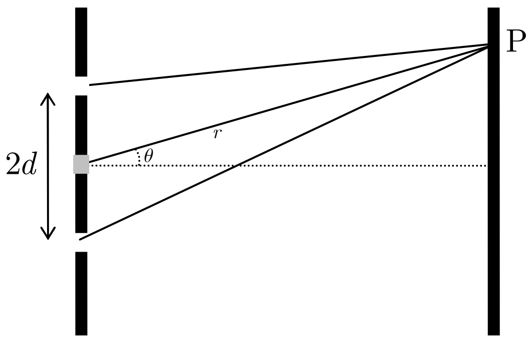
Question 7 (Tripos 2009)

The midpoints of two identical narrow, vertical slits are positioned at A and C, a distance $2d$ apart. A third, identical slit is positioned at B, midway between A and C. This slit contains an optical element which introduces a phase shift of π to any transmitted light. The three slits are illuminated by a plane wave of wavelength λ, which is incident normally on the plane of the slits. The intensity of the resulting wave, $I(\theta)$, is observed at a distant point P. The position vector of P from B subtends an angle θ with the perpendicular bisector of the line AC.

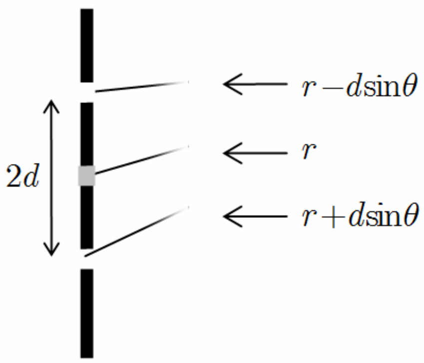
Obtain an expression for $I(\theta)\,/\,I(0)$ and sketch it, marking the angular positions and amplitudes of the important features. Illustrate your answers using phasor diagrams.

Solution:

Consider the point P described in the question. The situation is as follows:



Let the path length from middle slit to P be r, and consider the path lengths of the other beams above:



This means that the three waves above will have the form

$$\psi_{\mathrm{A}} = A\exp\left(i\left[\omega t - k\left\{r + d\sin\theta\right\}\right]\right)$$
$$\psi_{\mathrm{B}} = A\exp\left(i\left[\omega t - kr + \pi\right]\right)$$
$$\psi_{\mathrm{C}} = A\exp\left(i\left[\omega t - k\left\{r - d\sin\theta\right\}\right]\right)$$

A few points:

- The amplitudes are all identical, because the optical element does not affect the intensity of the light.
- We've taken the phase shift of π into account when dealing with the second wave.
- We assume the phase shift was $+\pi$ and not $-\pi$. Of course, in this case, it doesn't matter, because moving half a wavelength forward is the same as moving half a wavelength backwards. However, in some cases, it might, and the reasons why I knew it was a "+" and not a "–" is because the optical element can only shift the phase by "slowing down" the wave. Hence the "+".

The total resultant wave at the screen is then

$$\psi_{\mathrm{P}} = \psi_{\mathrm{A}} + \psi_{B} + \psi_{C} = A\,\mathrm{e}^{i\omega t}\,e^{-ikr}\left\{\exp\left(-ikd\sin\theta\right) + \exp\left(i\pi\right) + \exp\left(ikd\sin\theta\right)\right\}$$

Applying the same logic as in the last question, we can group all the pre-factors together:

$$\psi_{\mathrm{P}} = A_0\left\{\exp\left(-ikd\sin\theta\right) + \exp\left(i\pi\right) + \exp\left(ikd\sin\theta\right)\right\}$$

We then remember that $e^{i\pi} = -1$ and recognize a hyperbolic function:

$$\psi_{\mathrm{P}} = A_0\left\{2\cosh(ikd\sin\theta) - 1\right\}$$

We then remember that $\cosh ix = \cos x$

$$\psi_{\mathrm{P}} = A_0\left\{2\cos(kd\sin\theta) - 1\right\}$$

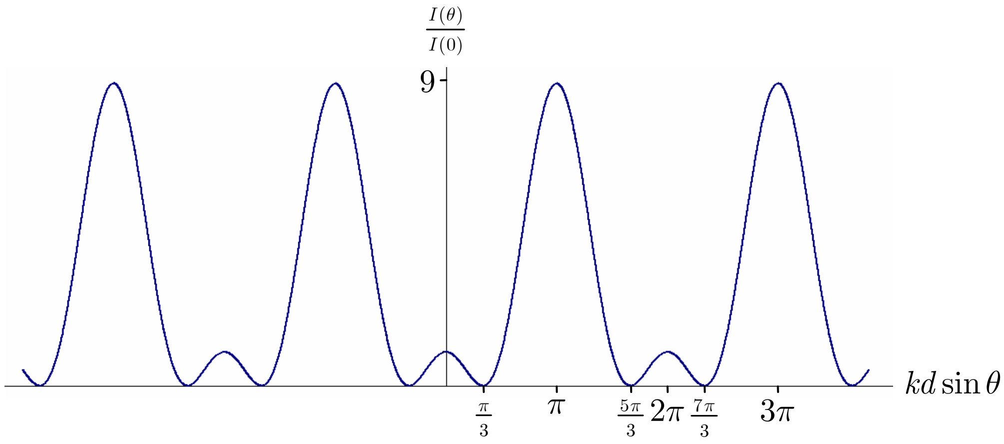
Intensity is amplitude squared, so

$$I(\theta) = A_0^2\left\{2\cos(kd\sin\theta) - 1\right\}^2$$

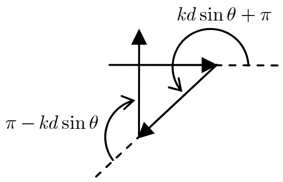
And

$$\boxed{\frac{I(\theta)}{I(0)} = \left\{2\cos(kd\sin\theta) - 1\right\}^2}$$

To plot it, simply plot a cosine function of amplitude 2, shift it down 1 notch and square the result. The graph looks like this:

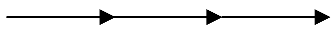


The phasor diagram here contains only three vectors. In general, it might look like this:



Clearly, we have three different kinds of points on our graph:

- **Big maxima**: the three vectors above lie in a straight line:

  This clearly occurs when the first exterior angle is an integer multiple of 2π, so

  $$kd\sin\theta+\pi=2n\pi$$
  $$\boxed{kd\sin\theta=(2n-1)\pi}$$

  In other words, this occurs at odd multiples of π; exactly what we observe from the formula above.

- **Small maxima**: the first two vectors point against each other, and the third vector points forward:

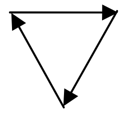
  

  This clearly occurs when the first exterior angle is an odd multiple of 2π, so

  $$kd\sin\theta+\pi=(2n-1)\pi$$
  $$kd\sin\theta=(2n-2)\pi$$
  $$\boxed{kd\sin\theta=2n\pi}$$

  In other words, this occurs at even multiples of π; exactly what we observe in the graph above.

- **Minima**: the three vectors above form a "triangle":



It's a bit difficult to find the right condition here. We can't just consider the sum of exterior angles, because there's stuff doubling back on itself and we don't know the last exterior angle.

Instead, we realize that *each* of the exterior angles of this triangle must be equal to $\frac{2\pi}{3} + 2n\pi$, where n is any number from 0 upwards (because adding 2π to an angle gets it back to the original place). So:

$$\left|\pi - kd\sin\theta\right| = \frac{2\pi}{3} + 2n\pi$$

Note the absolute value sign; the triangle could be reversed. We can consider both cases (down triangle and up triangle) separately to get the following

$$kd\sin\theta = \frac{(1-6n)\pi}{3} \qquad kd\sin\theta = \frac{(5+6n)\pi}{3}$$

(Remember that n can take negative values). Together, these do indeed provide all the solutions we had above: $\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \cdots$