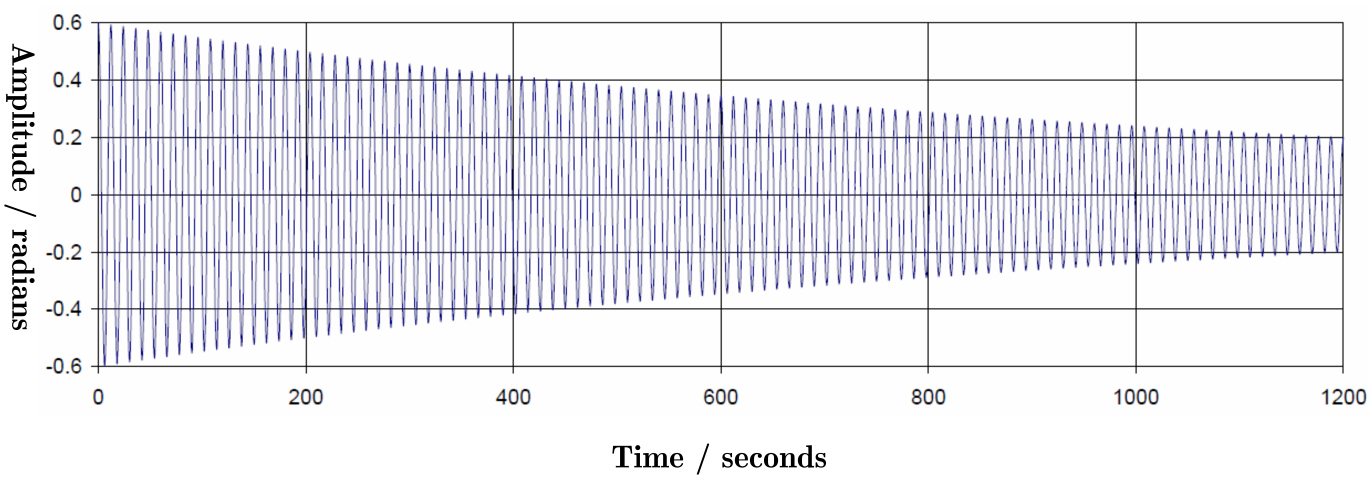
PHYSICS

*Answer **two** questions – you are advised to spend roughly 30 minutes on each question*

Question 1 Discuss the quality factor Q in damped oscillating systems; briefly explain how Q relates to relevant physical constants, and suggest how Q may be measured in two different ways. [4]

An antique clock keeps time by means of a torsional oscillator: the torsional bob has moment of inertia $I = 5.9 \times 10^{-5}$ kg m^2 and is suspended from a thin wire which provides a restoring force when the bob is rotated away from its equilibrium orientation. The system is very lightly damped. A free (undriven) oscillation of the rotation bob vs. time is shown in the graph below



Estimate angular frequency of the oscillation and the restoring torque per unit angular displacement (torsional spring constant) of the wire. [4]

Estimate the quality factor of this oscillation. Briefly suggest why the quality factor might be a good measure of how accurate you might expect the clock to be when the mechanism is driven at its resonant frequency. [3]

Explain whether you would expect the clock to run fast or slow when it is placed at the centre of a light turntable which is free to rotate on a frictionless bearing. What additional piece(s) of information would you require to calculate how much faster or slower? [4]

Newton's second law for a rotating body is $\tau = I\alpha$ where τ is the torque on the body, I is the body's moment of inertia and α is the body's angular acceleration

A torsional spring (wire) exerts a torque $\tau = \kappa\theta$ on a body attached to it, where θ is the angular displacement of the body from equilibrium, and κ is the torsional spring constant of the wire

Solution 1 The quality factor is a measure of how fast oscillations decay in a damped oscillator. A general damped oscillator has the following equation of motion

$$m\ddot{x} + b\dot{x} + kx = 0$$

This can be re-written as

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \qquad \gamma = \frac{b}{2m} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

The quality factor is defined as

$$Q = \frac{\omega_0}{2\gamma} = \frac{m}{b}\sqrt{\frac{k}{m}} = \frac{\sqrt{mk}}{b}$$ [**2**]

Two ways in which the Q-factor can be measured experimentally are

- By noting that the decay in amplitude per cycle of an oscillation is given by $e^{-2\pi\gamma/\omega_1}$, where ω_1 is the frequency of the damped system. When the oscillations are very lightly damped, $\omega_0 \approx \omega_1$, where ω_0 is the frequency of the un-damped system, and so the decay in amplitude per cycle of an oscillation is $e^{-2\pi\gamma/\omega_0} = e^{-\pi/Q}$.

  We can therefore observe the damped oscillator decay over time and use information about how fast it decays to work out Q. [**1**]
- By noting that Q is the ratio of the resonant frequency to the bandwidth of the oscillation, $Q = \omega_0 / \Delta\omega$.

  We can therefore force the oscillator at a range of frequencies and use the amplitude responses obtained to work out the resonant frequency and the bandwidth. This can be used to find Q. [**1**]

To find the angular frequency, we note from the graph that in the first 600 seconds, the system undergoes 50 oscillations. Therefore, the period is given by

$$T = \frac{600}{50} = 12 \text{ s}$$

We then have that

$$\omega = \frac{2\pi}{T} = 0.52 \text{ s}^{-1}$$ [**1**]

Now, imagine that the system has torsional spring constant κ. This means that when the bob is rotated an amount θ from equilibrium, the restoring torque is $\kappa\theta$. Since the system is very lightly damped, the equation of motion is approximately

$$I\ddot{\theta} \approx -\kappa\theta \Rightarrow \ddot{\theta} \approx -\frac{\kappa}{I}\theta$$

This is the equation for simple harmonic motion with angular frequency

$$\omega \approx \sqrt{\kappa / I}$$ [**2**]

The question gives us I and we found ω, so we can work out κ:

$$\kappa \approx \omega^2 I = 1.62 \times 10^{-5} \text{ N m rad}^{-1}$$ [**1**]

To find the quality factor, we note that in 1200 seconds = 100 oscillations, the amplitude decays from 0.6 to 0.2. Since the amplitude decays by $e^{-\pi/Q}$ per cycle, this implies that

$$0.6 \times \left(e^{-\pi/Q}\right)^{100} = 0.2$$

$$e^{-100\pi/Q} = \frac{1}{3}$$

$$-\frac{100\pi}{Q} = -\ln 3$$

$$Q = \frac{100\pi}{\ln 3} = 286$$ [**2**]

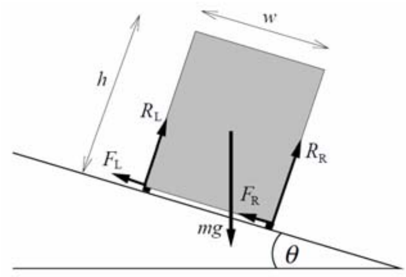
The question of accuracy is a difficult one, and can be approached from one of two angles. Either the following arguments would have got the mark: [**1**[

- Presumably, the driving mechanism somehow disturbs the bob's oscillation and therefore the accuracy of the clock. The larger Q, the slower the energy decay and the less often the driving mechanism will need to come into play. Therefore, the larger Q the less disturbed the oscillations, and the more accurate the clock.
- The larger the bandwidth of the oscillations the more likely it is for the pendulum to be driven slightly faster or slower than ω_0, resulting in inaccurate time keeping. The larger Q the smaller the bandwidth, and the more accurate the clock.

Putting the clock on a frictionless turntable amounts to putting the clock in outer space – the outer casing can now also rotate. By analogy to a linear system, the original system is equivalent to having two masses on either end of a spring with one mass fixed. Putting the clock in outer space is equivalent to releasing the second mass [**1**]. In the linear system, we know that the effect of releasing the second mass is to replace the mass of the system by the **reduced mass** $\mu = m_1 m_2 / (m_1 + m_2)$. A quick calculation shows that this is *less* than either the masses m_1 and m_2 [**1**]. By analogy, the net effect of putting the clock on the turntable is to *reduce* I, which translates to an **increase** in ω (since $\omega^2 = \kappa / I$). An increase in ω implies a decrease in T, which means that each oscillation will take **less time**. The clock will therefore run **fast** [**1**]. Once again by analogy, the quantity we would need to find the "reduced moment of inertia" is the moment of inertia of the outer casing [**1**].

Question 2 State how the forces on a body affect both its linear and rotational motion, and give the conditions for a body to be in equilibrium. [3]

A uniform rectangular block of mass m, width w and height h, which has small ridges on two of its bottom edges, as shown, is placed on a rough plane inclined at an angle θ to the horizontal.



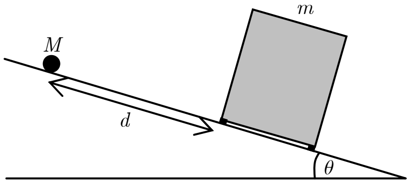
The forces acting on the block are its weight, the normal reaction forces R_L and R_R and the frictional forces F_L and F_R at the left and right ridges respectively, as shown. The coefficient of static friction between each of the ridges and the inclined plane is μ, and the masses and thicknesses of the ridges are negligible. How are the forces acting on the block related when the block is in equilibrium? [3]

Show that, as θ is increased,

(a) either the block will slide when $\tan\theta > \mu$ [*Hint*: consider the *total* frictional force and *total* reaction force on the block]

(b) or the block will tip over when $\tan\theta > w / h$ [4]

A point-like piece of ice of mass M is released from rest on the plane at a distance d from the block, slides freely on the surface, collides with the block and comes to rest as a result. The collision is observed to last 0.5 seconds (in other words, the ice touches the block, squishes, and then comes to rest, and the whole process takes 0.5 seconds from the time the ice first touches the block to the time the ice comes to rest). The coefficient of friction between the ice and the surface is 0.



Given that the block does *not* slide as a result of the collision, show that [*Hint*: once again, consider the *total* frictional force on the block]

$$d < \frac{m^2 g\left(\mu\cos\theta - \sin\theta\right)^2}{8M^2\sin\theta}$$ [5]

Solution 2 Let the forces on the body be denoted by $\boldsymbol{F_i}$ and the position vector of the point at which they act be $\boldsymbol{x_i}$ (with respect to the centre of mass of the body). Then the linear and rotational accelerations of the body are given by

- (Linear) $\boldsymbol{a} = \frac{1}{m}\sum \boldsymbol{F}_i$ (linear) **[1]**
- (Rotational) $\boldsymbol{\alpha} = \frac{1}{I}\sum \boldsymbol{x}_i \times \boldsymbol{F}_i$, where I is the moment of inertia of the body about its centre of mass **[1]**

For a body to be in equilibrium (both linear and rotational), we require

$$\sum \boldsymbol{F}_i = 0 \qquad\qquad \sum \boldsymbol{x}_i \times \boldsymbol{F}_i = 0$$ **[1]**

When the block is in equilibrium, the following must be true:

- **Forces parallel to the plane must balance**

$$F_L + F_R = mg\sin\theta \qquad (1)$$ **[1]**

- **Forces perpendicular to the plane must balance**

$$R_L + R_R = mg\cos\theta \qquad (2)$$ **[1]**

- **Turning moments with respect to the centre of the block must balance**

$$F_L\frac{h}{2} + R_L\frac{w}{2} + F_R\frac{h}{2} = R_R\frac{w}{2}$$
$$\left(F_L + F_R\right)h + R_L w = R_R w \qquad (3)$$ **[1]**

As the plane gets steeper, F will need to increase at both ridges to keep the block stationary. The block will slip if we ever have

$$\left(F_L + F_R\right) > \mu\left(R_L + R_R\right)$$ **[1]**

Using (1) and (2), we get

$$mg\sin\theta > mg\mu\cos\theta$$
$$\tan\theta > \mu$$ **[1]**

As the plane gets steeper, R_L decreases. The block tips when $R_L < 0$. **[1]**

Let's use our three equations to find expressions for R_L. First use (1) and (3) to eliminate F:

$$R_L + R_R = mg\cos\theta \qquad (2)$$
$$mgh\sin\theta + R_L w = R_R w \qquad (3)$$

Then use (2) to eliminate R_R

$$mgh\sin\theta + R_{\text{L}}w = mgw\cos\theta - R_{\text{L}}w$$
$$R_{\text{L}} = \frac{mgw\cos\theta - mgh\sin\theta}{2w}$$

So the block tips when

$$R_{\text{L}} = \frac{mgw\cos\theta - mgh\sin\theta}{2w} < 0$$
$$\tan\theta > \frac{w}{h}$$ **[1]**

From the time the piece of ice is released to the time it collides with the block, it loses potential energy

$$Mgh = Mgd\sin\theta$$

This is equal to the kinetic energy gained by the ice. The velocity of the ice immediately before the collision is therefore

$$v = \sqrt{2gd\sin\theta}$$ **[1]**

The ice is brought to rest by the collision. The change in momentum of the ice during the collision is therefore $Mv = M\sqrt{2gd\sin\theta}$. This is equal to the impulse exerted by the block on the ice. By Newton's Third Law, this is equal to the impulse exerted by the ice on the block

$$\text{Impulse} = M\sqrt{2gd\sin\theta}$$ **[1]**

The collision lasts 0.5 seconds. Assuming that the force exerted by the ice on the block (and vice versa) involved is constant throughout, we have

$$\text{Impulse} = \int \mathcal{F}\ \mathrm{d}t = \mathcal{F} \times 0.5 = M\sqrt{2gd\sin\theta}$$
$$\mathcal{F} = M\sqrt{8gd\sin\theta}$$ **[1]**

The maximum combined force of friction that can be provided by the front and back ridges of the block is $2F_{\text{max}} = \mu\left(R_L + R_R\right)$. For the block to remain stationary after the collision, we require

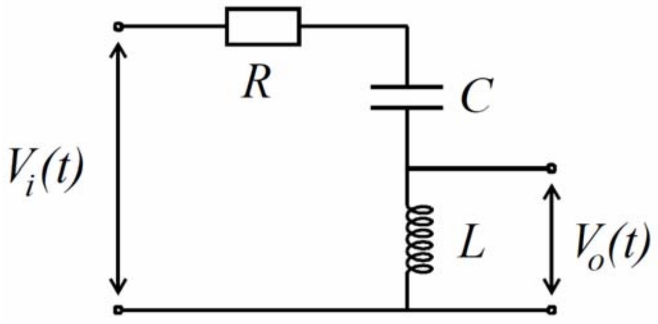
$$mg\sin\theta + \mathcal{F} < 2F_{\text{max}}$$ **[1]**
$$mg\sin\theta + M\sqrt{8gd\sin\theta} < \mu\left(R_L + R_R\right)$$

Using equation (2) above, this becomes

$$mg\sin\theta + M\sqrt{8gd\sin\theta} < \mu mg\cos\theta$$
$$d < \frac{m^2 g\left(\mu\cos\theta - \sin\theta\right)^2}{8M^2\sin\theta}$$ **[1]**

Question 3 Define the term impedance as it relates to AC circuits and derive the expressions for the complex impedances of a capacitor and an inductor, $Z_C = 1/(i\omega C)$ and $Z_L = i\omega L$ respectively. [5]

In the circuit below, the voltage $V_i(t) = V_1 \,\mathrm{Re}\{e^{i\omega t}\}$ produces a current $I(t) = I_1 \,\mathrm{Re}\{e^{i(\omega t+\phi)}\}$. Determine I_1 and ϕ. [5]



The output voltage across the inductor is $V_0(t) = V_2 \,\mathrm{Re}\{e^{i(\omega t+\theta)}\}$. Determine V_2 and θ in terms of V_1, R, C, L and ω. Sketch graphs showing the dependence of V_2 and θ on ω. Indicate on your graph calculated values of V_2 and θ at $\omega = 0$, $\omega = (LC)^{-1/2}$ and the asymptotic values as $\omega \to \infty$. [5]

Solution 3 **Impedance** describes a measure of opposition to alternative current (AC). Impedance extends the concept of resistant to AC circuits, describing not only the relative amplitudes of the voltage and current, but also the relative phases.

Mathematically, we define the impedance Z as the voltage-current ratio when the voltage is a single complex exponential at a particular frequency ω:

$$Z = \frac{V(t)}{I(t)} \qquad\qquad V(t) = V_0 e^{i\omega t}$$ **[1]**

Consider a capacitor driven so that the potential difference between its two ends is $V(t) = V_0 e^{i\omega t}$. We know that the charge stored in a capacitor of capacitance C is related to the potential difference across its ends by the expression $q = CV$. The current through the capacitor is therefore given by

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} = C\frac{\mathrm{d}V}{\mathrm{d}t} = CV_0 i\omega e^{i\omega t}$$

The impedance is therefore given by

$$Z = \frac{V(t)}{I(t)} = \frac{V_0 e^{i\omega t}}{CV_0 i\omega e^{i\omega t}} = \frac{1}{i\omega C}$$ **[2]**

Consider a capacitor driven so that the potential difference between its two ends is $V(t) = V_0 e^{i\omega t}$. We know that the voltage across and current through an inductor are related by $V = L(\mathrm{d}I / \mathrm{d}t)$. Therefore

$$L\frac{\mathrm{d}I}{\mathrm{d}t} = V_0 e^{i\omega t}$$

$$I(t) = \frac{V_0}{L}\int e^{i\omega t}\ \mathrm{d}t$$

$$I(t) = \frac{V_0}{i\omega L}e^{i\omega t}$$

The impedance is therefore given by

$$Z = \frac{V(t)}{I(t)} = \frac{V_0 e^{i\omega t}}{(V_0 / [i\omega L])e^{i\omega t}} = i\omega L$$ **[2]**

Impedances in series *add*. Therefore, the total impedance of the circuit in the question is

$$Z_{\text{tot}} = R + i\omega L + \frac{1}{i\omega C}$$ **[1]**

$$Z_{\text{tot}} = R + \left(\omega L - \frac{1}{\omega C}\right) i$$

$$Z_{\text{tot}} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \exp\left[i \arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)\right] \quad \textbf{[1]}$$

The current is given by

$$I = \frac{V}{Z_{\text{tot}}} = \frac{V_1 e^{i\omega t}}{Z_{\text{tot}}} \quad \textbf{[1]}$$

$$I = V_1 \exp\left[i\omega t\right] \frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \exp\left[-i \arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)\right]$$

$$I = \frac{V_1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \exp\left[i\left(\omega t - \arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)\right)\right]$$

Therefore, we have

$$I_1 = \frac{V_1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \quad \textbf{[1]}$$

$$\phi = -\arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right) \quad \textbf{[1]}$$

Since the circuit is in series, the current through the inductor is $I(t)$ as above. We then have

$$V_0(t) = I(t) Z_{\text{inductor}}$$

$$V_0(t) = I_1 \omega L e^{i(\omega t + \phi)} e^{i(\pi/2)}$$

And so

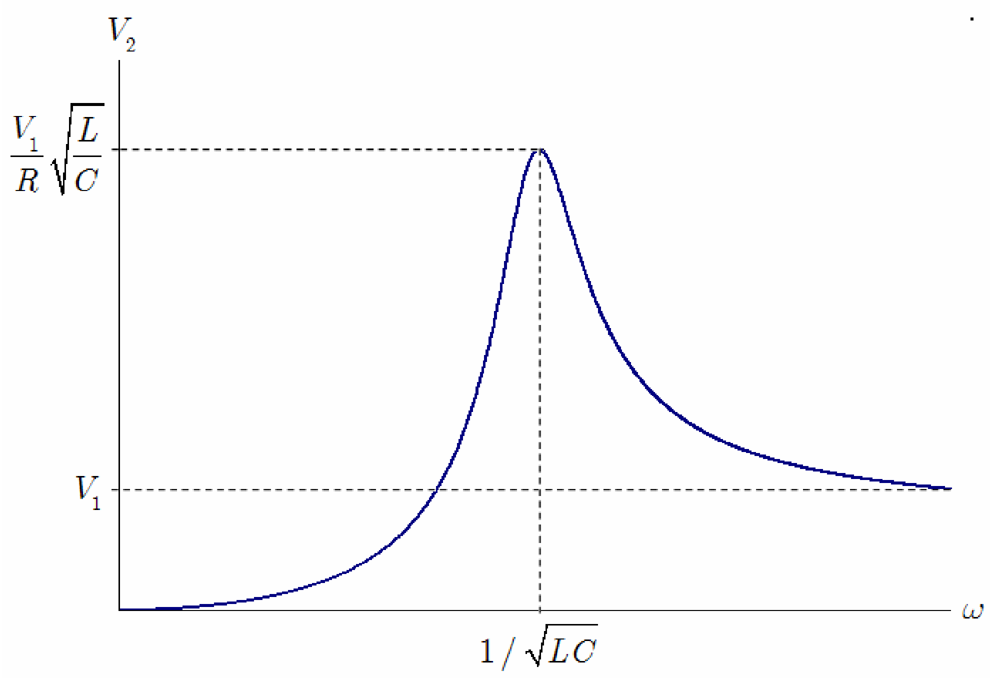
$$V_2 = \frac{V_1 \omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$\theta = \frac{\pi}{2} - \arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)$$

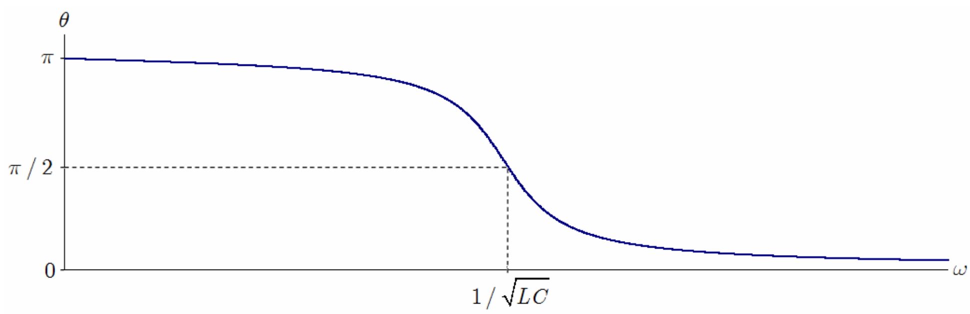
Let's first find the value of these at the three values of ω specified:

	$\omega = 0$	$\omega = 1/\sqrt{LC}$	$\omega \to \infty$	
V_2	0	$\frac{V_1}{R}\sqrt{\frac{L}{C}}$	V_1	**[1]**
θ	π	$\pi/2$	0	**[1]**

The graphs look as follows:



[2]



[1]

It is interesting (though not necessary to get the marks) to realise what is going on at each of the special values of ω above:

- **At** $\omega = 0$, the inductor simply acts as a piece of wire (very low impedance), and the capacitor almost acts as a short (very high impedance). Therefore:
  - All the potential difference is dropped across the capacitor, and so no potential drops across the inductor. It makes sense for the potential difference across the inductor to be 0.
  - The behaviour in the circuit is dominated by the capacitor, so current in the circuit **leads** V_i by $\pi/2$. Across the inductor, however the current **lags** behind V_0 by $\pi/2$. Thus, V_i **lags** V_0 by $\frac{\pi}{2} + \frac{\pi}{2} = \pi$.

- **At** $\omega = 1/\sqrt{LC}$, the inductor and capacitor both have the *same* impedance.
    - The current is in phase with V_i because the *leading* effect of the capacitor and the *lagging* effect of the inductor cancel each other out exactly at resonance. However, across the inductor, the current still **lags** behind V_0 by $\pi / 2$. Thus, V_i **lags** V_0 by $\pi / 2$.
    - (The value of V_2 here is slightly more complicated, so I won't spend half a page doing it).
- **As** $\omega \to \infty$, the capacitor simply acts as a piece of wire (very low impedance) but the inductor almost acts as a short (very high impedance):
    - All the potential V_1 across the circuit is dropped across the inductor.
    - The behaviour in the circuit is dominated by the inductor, so current in the circuit **lags** V_i by $\pi / 2$. Across the inductor, the current also **lags** behind V_0 by $\pi / 2$. These two therefore cancel, and V_i is in phase with V_0.