Waves

Generalities

Basics

- A **wave** is the means by which **information** about a **disturbance** in **one place** is carried to another **without bulk translation** of the **intervening medium**.
- To derive the wave equation, proceed as follows:
  - Consider a **wave** travelling in the **positive *x*** direction such that at ***t* = *0***, the function ***f(x)*** describes the shape of the wave, such that
  $$\psi(x,0) = f(x)$$
  - After time ***t*** has elapsed, the wave will have moved a distance ***vt*** to the **right** (where ***v*** is the **velocity** of the wave). As such
  $$\psi(x,t) = f(x - vt) = \psi(x - vt, 0)$$
  - If we let ***u* = *x* − *vt***, then we have
  $$\psi(x,t) = f(u)$$
  And by the **chain rule**
  $$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}u}$$
  $$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)\frac{\partial u}{\partial x} = \frac{\mathrm{d}^2 f}{\mathrm{d}u^2}$$
  Similarly:
  $$\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial t} = -v\frac{\mathrm{d}f}{\mathrm{d}u}$$
  $$\Rightarrow \frac{\partial^2 \psi}{\partial t^2} = v^2\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}$$
  - Equating our terms for $\mathrm{d}^2 f / \mathrm{d}u^2$:
  $$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$
- Important notes on the wave equation:
  - It applies to *any* sort of wave motion, of any form – we have not specified the form of *f*.

  - It is **linear** in ψ, which means that the **principle of superposition** applies.
- In general, we consider **harmonic waves**. Any other waveform can be **Fourier analysed** into a number of such harmonic waves. Harmonic waves take the form

$$\boxed{\psi(x,t) = A\,\mathrm{Re}\left(e^{i(\omega t - kx)}\right)}$$

Where $(\omega t - kx)$ is the **phase** of the wave, and

$$\boxed{k = \frac{\omega}{v_p} = \frac{2\pi}{\lambda}}$$

Each of the two parts of this equality arise as follows:

  - k is the rate at which the phase changes with position. ω is the rate at which the phase changes with time, and v_p is the rate at which it moves with time. The first equality follows.
  - At fixed time t, the phase of the wave changes by 2π when the distance changes by λ – the second equality follows.

And

$$\boxed{\omega = 2\pi\nu}$$

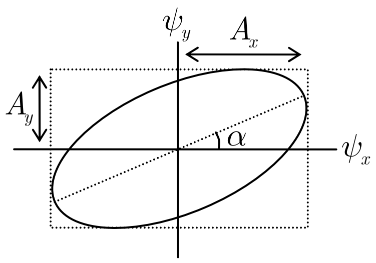
Polarisation

- In **transverse** waves, the displacement of the medium is **perpendicular** to the **direction of motion** of the wave. There are therefore **two orthogonal directions** along which the displacement can take place. The **amplitude** and **polarisation** of the waves along these two directions define the **polarisation** of the wave.
- Consider a wave travelling in the z direction. The x and y components of the wave take the form

$$\psi_y = A_y e^{i(\omega t - kz)}$$
$$\psi_x = A_x e^{i(\omega t - kz + \phi)}$$

The wave can then be polarised in a number of ways.

- ***Linear polarisation*** – every point on the string oscillates parallel to a given line. We have:
    - $\phi = 0$ or an integer multiple of π.
    - The wave oscillates with amplitude $A = \sqrt{A_x^2 + A_y^2}$ at an angle $\arctan\left(A_y / A_x\right)$ to the x axis.
- ***Circular polarisation*** – sometimes, the displacement will follow a **circular path** in the x-y plane. For this to happen,
    - $\phi = \pi / 2$
    - $A_x = A_y$
- ***Elliptical polarisation*** – this is the most general way a wave can be polarised, for any parameters. The displacement follows an **elliptical path** in the x-y plane, as follows:



Where

$$\tan 2\alpha = \frac{2A_y A_x \cos\phi}{A_y^2 + A_x^2}$$

Impedance

- The concept of **wave impedance** is used to define the relationship between the **force** and the **wave response**:

$$\text{Impedance} = \frac{\text{driving force}}{\text{velocity response}}$$

  It is extremely important to note that the **velocity response** is the **rate of change of displacement** **<u>in the medium</u>**, <u>*not*</u> the speed of propagation of the wave.
- The **power** fed into a wave is given by the **product** of the **driving force** and **velocity response**. As such

- If the impedance is **real**, the force and velocity are **in phase** and the power input is **maximised**.
- If the impedance is **purely imaginary** (eg: below the cut-off frequency in plasmas and waveguides), **no energy** can be fed into the medium.

- We can use the impedance to find the average power input into the wave. Let **u** be the transverse velocity. For an **oscillator**, the **mean power input** is given by

$$P_{av} = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{F}\boldsymbol{u}^*\right)$$

Given that $\boldsymbol{F} = \boldsymbol{Z}\boldsymbol{u}$:

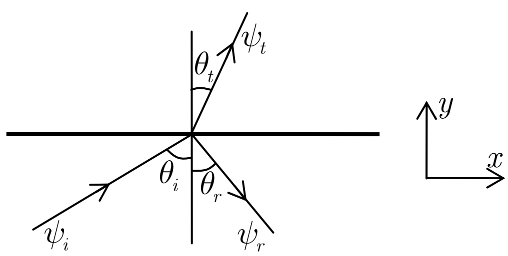
$$\begin{aligned} P_{av} &= \frac{1}{2}\mathrm{Re}\left(\boldsymbol{Z}\boldsymbol{u}\boldsymbol{u}^*\right) \\ &= \frac{1}{2}|\boldsymbol{u}|^2\,\mathrm{Re}(\boldsymbol{Z}) \\ &= \frac{1}{2}\left|\dot{\boldsymbol{\psi}}\right|^2\mathrm{Re}(\boldsymbol{Z}) \end{aligned}$$

Assuming that $\boldsymbol{Z}$ is real, and that the wave is harmonic with frequency ω, we therefore get

$$\text{Mean power} = \frac{1}{2}Z\omega^2 A_0^2$$

Reflection and Transmission

- Whenever a wave encounters a **change** in **impedance** in the medium though which it is travelling, some of its energy will be **reflected** and some will be **transmitted**.
- Consider a wave travelling in one medium incident on another medium, as follows:



Assume that the waves are polarised in the x–y plane, and that the waves all have the form

$$\boldsymbol{\psi}_i = \boldsymbol{\psi}_{i0} e^{i(\boldsymbol{k}_i \cdot \boldsymbol{r} - \omega_i t)}$$

$$\boldsymbol{\psi}_r = \boldsymbol{\psi}_{r0} e^{i(\boldsymbol{k}_r \cdot \boldsymbol{r} - \omega_r t)}$$

$$\boldsymbol{\psi}_t = \boldsymbol{\psi}_{t0} e^{i(\boldsymbol{k}_t \cdot \boldsymbol{r} - \omega_t t)}$$

- We can go through a process called **phase-matching**. At all times, the component of the waves **parallel** to the interface (the x component) must be continuous across the boundary – therefore:

$$\psi_{i,x} - \psi_{r,x} = \psi_{t,x}$$

$$\psi_{i0,x} e^{i(k_i x \sin\theta_i - \omega_i t)} \cos\theta_i - \psi_{r0,x} e^{i(k_r x \sin\theta_r - \omega_r t)} \cos\theta_r = \psi_{t0,x} e^{i(k_t x \sin\theta_t - \omega_t t)} \cos\theta_t$$

Because these must be true for all values of $\boldsymbol{r}$ and t, we must have that

$$\omega_i = \omega_r = \omega_t$$

$$k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$$

We now simply use the fact that $k = \omega / v_p$ to deduce that, since all the ω are equal and the wave speed *must* be the same for the incident and reflected wave (since they travel in the same medium)

$$\theta_i = \theta_r$$

$$\Rightarrow k_i = k_r$$

And therefore

$$\frac{\sin\theta_t}{\sin\theta_r} = \frac{k_r}{k_t}$$

This is **Snell's Law**

- The results obtained using phase matching apply to all waves reflecting at a boundary. However, to deduce any results regarding the *amplitude* of the various waves, information is required as to the boundary conditions at the interface, which will necessarily be different for different kinds of waves.
- A further complication results from the polarisation of the wave involved – waves polarised *perpendicular* to the x–y plane will have a parallel component

independent of θ_i, whereas those polarised *parallel* to the plane will not.

- In the simple case, however, in which $\theta_i = 0$, the following two equations are obtained:
    - The first one involves the continuity of the wave at the boundary.

$$\psi_i + \psi_r = \psi_t$$

    - The second one involves the continuity of *one* other quantity at the boundary. Usually, that quantity is the "force", or other equivalent quantity. This therefore yields

$$Z_1\psi_i' - Z_1\psi_r' = Z_2\psi_t'$$

      Integrating:

$$Z_1\psi_i - Z_1\psi_r = Z_2\psi_t$$

      Where Z_1 and Z_2 are the impedances of the media through which the wave is travelling.

Solving these equations, we find that the amplitude transmission and reflection coefficients are

$$r = \frac{\psi_r}{\psi_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

$$t = \frac{\psi_t}{\psi_i} = \frac{2Z_1}{Z_1 + Z_2}$$

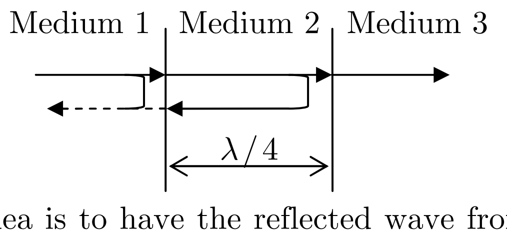
A few notes:

- These expressions are for the *amplitude* term. When dealing with electromagnetic radiation, they must be **squared** to obtain the intensity coefficient.
- Similarly, if one needs expressions for the "force" term, then the impedances in the expressions above each need to be reciprocated (ie: $Z_i \to 1/Z_i$), to reflect the fact that $1/Z$ terms will appear in the "displacement matching" equations.

  - If either the impedances are complex, then there are phase differences between the incident, reflected and transmitted waves.
- We showed, above, that the rate at which energy is transferred in a harmonic wave is given by

$$\frac{1}{2}Z\omega^2A^2$$

Using these results and r and t above, it is trivial to work out the ratios of transmitted and reflected energies at the boundary. If the impedances are complex, the alternative expression $\frac{1}{2}\mathrm{Re}(Z)\omega^2A^2$ must be used.
- Often, we need to transmit waves from one medium to another (or to extract energy at the boundary of the medium in which the wave is travelling), but we want to make sure as little reflects back as possible. To do this, we want to try as hard as possible to match the impedances as well as possible. Two particular examples of how this is done are as follows:
  - In optics, we want to design lenses that reflect as little light as possible. Here, it is difficult to match the impedance of air and glass, but what we *can* do is use a little trick called a "$\lambda/4$ coupler":



The idea is to have the reflected wave from the 2–3 interface interfere destructively with the reflected wave at the 1–2 interface, giving no reflection. It's reasonably obvious that the width of the intervening layer should be $\lambda/4$, but it's harder to see that $Z_2 = \sqrt{Z_1Z_3}$. This is used on camera lenses (a technique known as "blooming"), but total removal of reflection can only work for *one* wavelength, which explains the purplish glow from many camera lenses.

- Another technique involves the *gradual* change of impedance from one medium to another. The reflected waves at each different boundary will have different phases, and will cancel each other out. An example of such a technique is that of gently widening horns in old gramophones and clarinets, going from the inside of the instrument to the air outside.

3D Waves

- To define a wave in 3D, we need to specify the **shape** and **orientation** of the wavefronts. The simplest form of 3D wave is a **plane wave**, of **infinite extent**, and whose wavefronts are **planes** with *perpendicular* vector ***k***. The wave can then be written:

$$\psi(\boldsymbol{r},t) = Ae^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$$

  And the **wave equation** becomes:

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

- Another example is a **spherical wave**, wave by a point-disturbance at the **origin** ($\boldsymbol{r} = \mathbf{0}$):

$$\psi(\boldsymbol{r},t) = \frac{A}{r}e^{i(\omega t - k|\boldsymbol{r}|)}$$

  The intensity (proportional to ψ^2) decays as r^{-2}, thus conserving energy, since the **wavefronts** are **spheres** surrounding the origin.

Dispersive Waves

- So far, we have assumed that the speed of waves is **independent** of the frequency. This is true for **electromagnetic waves in a vacuum**. In other cases, though, the **wave speed** (or **phase velocity**), $v_p = \omega / k$, depends on the **frequency** of the wave. These waves are known as **dispersive waves**.

- Dispersion often arises from the fact that the medium through which the wave travels has a resonance at a particular frequency which can be excited by the passage of the wave.
- The way the **phase velocity** depends on **frequency** is called the **dispersion relation**:
  - In **normal dispersion**, waves of **higher frequency** travel **slower**. In other words, the gradient of a graph of ω again k gets **less**.
  - In **anomalous dispersion**, the situation is reversed.
- Important examples include:
  - **Optical waves** in media such as **glass** (*c.f.* Snell's Law and prisms).
  - **Surface waves** in **deep water**.
- An important example involves **wave packets**.
  - Infinite wavetrains are inherently featureless (even turning them "on" and "off" means that they're no longer infinite). All **real waves** (carrying **information**) have a beginning and an end – ie: their **amplitude varies with time**. Each "pulse" represents a **wave packet**.
  - To make up this **wave packet**, we need to **superimpose** waves with a **range of frequencies** and **wavenumbers** (Δk) [*cf*: Fourier analysis]. The sharper the "pulse", the more frequencies are needed.
  - It is, in fact, important to note that the **range of frequencies** in the wave packet is **inversely proportional** to the **packet width**. So:

$$\Delta \nu_{in\ pulse} = 1 / \Delta t_{of\ pulse}$$

  - In a **dispersive medium**, the **different components** will not travel at the same speed – the wavepacket will **spread out** and eventually **disappear**.
  - Another consequence is that the wavegroup itself (= the point of **maximum amplitude** in the wavegroup) will move at a speed different from the phase velocity. This **group velocity** is the speed at which energy moves through the medium, and so it's a crucial parameter.
- For the simple case of two superimposed waves, the difference between the group and phase velocity can be seen by combining two waves of frequency $\omega + \Delta\omega$ and $\omega - \Delta\omega$ and wavevectors $k + \Delta k$ and $k - \Delta k$. The envelope is seen to move at a group velocity $\Delta\omega / \Delta k$.
- In a more general case of wavegroups, we can determine the speed of the wavegroup (**group velocity**) by consider the speed at which the point of **maximum amplitude** moves. At time t, this corresponds to the point at which the **phase**, $\omega t - kx + \phi$ is **the same** for all waves (ie: of all ω). Therefore:

$$\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega t - kx + \phi) = 0$$

$$t - \frac{\mathrm{d}k}{\mathrm{d}\omega}x = 0$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{x}{t} = v_g$$

- We can easily find the relationship between the phase and group velocity as follows:

$$\omega = v_p k$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = v_g = v_p + k\frac{\mathrm{d}v_p}{\mathrm{d}k}$$

In terms of $\lambda = 2\pi / k$:

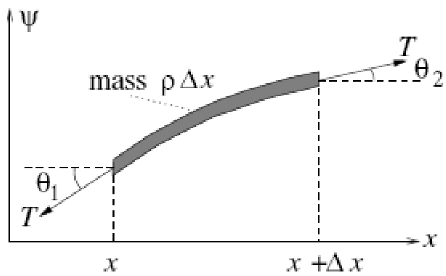
$$v_g = v_p + \frac{2\pi}{\lambda}\frac{\mathrm{d}v_p}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}k}$$
$$v_g = v_p - \frac{2\pi}{\lambda}\frac{\mathrm{d}v_p}{\mathrm{d}\lambda}\frac{2\pi}{k^2}$$
$$v_g = v_p - \frac{2\pi}{\lambda}\frac{\mathrm{d}v_p}{\mathrm{d}\lambda}\frac{\lambda^2}{2\pi}$$
$$v_g = v_p - \lambda\frac{\mathrm{d}v_p}{\mathrm{d}\lambda}$$

- Often, a much easier way of finding the group velocity from the phase velocity is to write the phase velocity in the form $kv_p = \omega$ and differentiating directly with respect to k.

Examples of Waves

Waves on a string

- Consider a small element of string of length Δx and mass $\rho\Delta x$, with longitudinal tension T:



The restoring force on the left-hand-side of the element is $T\sin\theta_1$, which, in the limit of small displacement is $T\tan\theta_1 = T\left.\frac{\partial\psi}{\partial x}\right|_x$. The total restoring force is therefore:

$$T\left(\left.\frac{\partial\psi}{\partial x}\right|_x - \left.\frac{\partial\psi}{\partial x}\right|_{x+\delta x}\right) = T\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)\Delta x = T\frac{\partial^2\psi}{\partial x^2}\Delta x$$

Using Newton's Second Law:

$$T\frac{\partial^2\psi}{\partial x^2}\Delta x = \frac{\partial^2\psi}{\partial t^2}\rho\Delta x$$
$$\frac{\partial^2\psi}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2\psi}{\partial t^2}$$

Which is the wave equation, with speed $v = \sqrt{T/\rho}$.

- We can now find an expression for the impedance of a wave on a string:
$$\text{Impedance} = \frac{\text{transverse driving force}}{\text{transverse velocity}}$$
In the case of a string,
$$\text{Transverse driving force} = -T\sin\theta \approx -T\frac{\partial\psi}{\partial x}$$
So:
$$\text{Impedance} = -\frac{T\dfrac{\partial\psi}{\partial x}}{\dfrac{\partial\psi}{\partial t}}$$
But when deriving the wave equation, we saw that, for a wave travelling in the $+ve$ x-direction:
$$\frac{\partial\psi}{\partial x} = -\frac{1}{v}\frac{\partial\psi}{\partial t}$$
And so:
$$\text{Impedance} = \frac{T}{v} = \sqrt{T\rho} = \rho v$$
For a wave travelling in the $-ve$ x direction, the impedance is simply given by $-T/v$.
- It is important, in such problems, to carefully consider the sign of the derivatives used.
- We can also formally rationalise the requirement that the force be continuous at the boundary. If we consider an infinitesimal mass element at the boundary, and if the forces were not matched, this infinitesimal element would have infinite acceleration – clearly impossible.
- The energy in the vibrating string, ranging from point a to point b, consists of two components:
    - The **kinetic energy**, given by
    $$KE = \int_a^b \frac{\rho\dot{y}^2}{2}\,\mathrm{d}x$$
    - The **potential energy** given by
    $$PE = \int_a^b \frac{Ty'^2}{2}\,\mathrm{d}x$$

A nice way to rationalise this expression is by considering a vibrating string, and noticing that the total extension in the string as a result of the vibrations is

$$\mathrm{d}S = \int_a^b \sqrt{1+y'^2} - 1 \; \mathrm{d}x$$

Carrying out a Binomial Expansion:

$$\mathrm{d}S = \frac{1}{2}\int_a^b y'^2 \; \mathrm{d}x$$

And therefore, **assuming the tension is constant throughout** (we would otherwise need an *Extension*^{2} component), the potential energy is given by:

$$\mathrm{d}U = \frac{T}{2}\int_a^b y'^2 \; \mathrm{d}x$$

We can use these expressions to obtain one for the **rate of change of energy** in the string (ie: the power transmitted):

$$E = \int_a^b \frac{\rho \dot{y}^2}{2} + \frac{Ty'^2}{2} \mathrm{d}x$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_a^b \rho\dot{y}\ddot{y} + Ty'\dot{y}' \,\mathrm{d}x$$

Using the **wave equation** ($Ty'' = \rho\ddot{y}$):

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_a^b Ty''\dot{y} + Ty'\dot{y}' \,\mathrm{d}x$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}(Ty'\dot{y}) \,\mathrm{d}x$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \left[(Ty')\dot{y}\right]_a^b = \boldsymbol{Power\ flow}$$

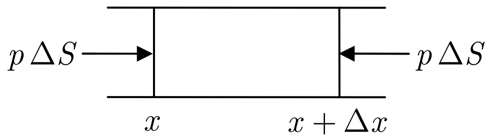
This is in the form of $\text{Stress} \times \text{Velocity}$, which is a recurring result in elastic systems.

Sound waves

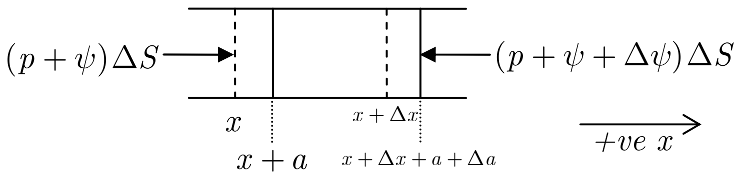
- Sound is a **longitudinal wave**. The direction of the displacement is along the direction of travel of the wave. Transmission occurs by **compression** and

rarefaction of the medium in which the wave is travelling.

- Consider an element in a column of gas, cross-section Δx, and at a pressure p when no wave is propagating:



Let's now consider a wave travelling through the medium, with displacement $a(x,t)$ and pressure $\psi(x,t)$:



- The **volume change** of the element is therefore given by:

$$(\Delta x+\Delta a)\Delta S-\Delta x\Delta S=\Delta S\Delta a=\Delta S\frac{\partial a}{\partial x}\Delta x$$

And the **fractional change in volume** is then given by

$$\begin{aligned}\frac{\Delta V}{V}&=\frac{\Delta S\frac{\partial a}{\partial x}\Delta x}{\Delta S\Delta x}\\&=\frac{\partial a}{\partial x}\end{aligned}$$

- Now, the **force** in the **positive *x*-direction** caused by the pressure imbalance in the wave is given by

$$\begin{aligned}&(p+\psi)\Delta S-(p+\psi+\Delta\psi)\Delta S\\&=-\Delta\psi\Delta S\\&=-\frac{\partial\psi}{\partial x}\Delta x\Delta S\end{aligned}$$

- We also note that in a sound wave, the pressure changes occurs so quickly that no heat is exchanged with the surroundings. The changes are **adiabatic**. As such, we have that

$$pV^{\gamma}=\text{constant}$$

With $\gamma=C_p/C_v$, and taking a value of 1.4 for air. Differentiating:

$$V^{\gamma}\mathrm{d}p + \gamma p V^{\gamma-1}\mathrm{d}V = 0$$

$$\mathrm{d}p = -\gamma p \frac{\Delta V}{V}$$

But $\mathrm{d}p$ is simply the pressure change associated with the wave passing, and $\Delta V / V$ is simply the fractional change in volume. As such:

$$\boxed{\psi = -\gamma p \frac{\partial a}{\partial x}}$$

Differentiating:

$$\frac{\partial \psi}{\partial x} = -\gamma p \frac{\partial^2 a}{\partial x^2}$$

- Feeding this in to our result for "force" on the element of gas, above:

$$\text{Force} = \gamma p \frac{\partial^2 a}{\partial x^2} \Delta x \Delta S$$

Applying Newton's Law to this element of gas, of mass $\rho \Delta x \Delta S$:

$$\gamma p \frac{\partial^2 a}{\partial x^2} \Delta x \Delta S = \rho \Delta x \Delta S \frac{\partial^2 a}{\partial t^2}$$

And we therefore get:

$$\boxed{\frac{\partial^2 a}{\partial x^2} = \frac{\rho}{\gamma p} \frac{\partial^2 a}{\partial t^2}}$$

- This is the **wave equation**, with

$$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma R T}{m}}$$

This implies that the speed of sound primarily depends on m, the molar mass. γ will also have an effect, but less so than m.

- Using the boxed relation above, we can also analyse sound waves in terms of **pressure variations** rather than in terms of **displacement variations**. For a **harmonic wave** $a = a_0 e^{i(\omega t - kx)}$, we have that

$$\psi = i\gamma p k a$$

In other words, the pressure *leads* the displacement by $\pi / 2$, and has amplitude $\gamma p k a_0$.

Sound waves in solids and liquids

- **Longitudinal** waves can also be supported in **solids** and **liquids**. In both cases, we need a relation between the **pressure** due to the wave and the **strain** in the medium:

$$\psi_p = -K\frac{\partial a}{\partial x}$$

Where K is the relevant modulus (in a gas, we found that it was equal to γp). This gives a velocity:

$$v = \sqrt{\frac{K}{\rho}}$$

- In **gasses** and **liquids**, even though the pressure is **isotropic**, the expansion and rarefaction only takes place **in the direction of passage of the wave**. Hence, the fractional volume change is directly proportional to the change in length of the element (ie: $\Delta V / V = \partial a / \partial x$). The relevant modulus is then the **bulk modulus**, B.
- In **solids**, things are more complicated, because when **longitudinal compression** occurs, the solid can also expand in the **transverse direction** (the ratio of the two strains is known as **Poisson's Ratio**). In **bulk solids**, where no sideways expansion is possible, a **larger** pressure is required to produce a longitudinal strain, giving a larger modulus. In any case, the relevant modulus is the **Young's Modulus** – the ratio of the **longitudinal stress** (pressure) to the **longitudinal strain**:

$$Y = \frac{\mathrm{d}p}{\partial a / \partial x}$$

- The **acoustic** impedance in such materials is also given by $Z = \rho v$.