Mechanics

- **By experiment**, we find that the **net external force** acting on a body is the **sum of the external forces acting on it**.
- For **static equilibrium** to be maintained, no matter where we cut a body or a system, we require:

$$\boxed{\mathbf{F}_{ext} = \sum_i \mathbf{F}_i = \mathbf{0}}$$

and we require, about **any point**

$$\boxed{\mathbf{G}_{ext} = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \mathbf{0}}$$

or, in scalar form: $G_{ext} = \sum_i d_i \left|\mathbf{F}_i\right| = 0$

Where:

- $\mathbf{r}_i$ is the **distance** of the **point of action** of the force $\mathbf{F}_i$ from a **given point**.
- d_i is the **PERPENDICULAR distance** of the **point of action** of the force $\mathbf{F}_i$ from a **given point**.

- **Newton's Laws are**:

1) "*Every body **continues** in its **state of rest**, or of **uniform motion** in a **straight line**, unless it is **compelled** to change by **forces** impressed upon it*".
2) "*The **rate of change of momentum** is **proportional** to the **net force applied**, and is in the **same direction** as the applied force*".

$$\boxed{\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\rho} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v})}$$

Where:

- $\mathbf{F}$ is the **force** applied.
- $\mathbf{v}$ is the **velocity** of the body.
- m is the **inertial mass** of the body. This is the same as the **gravitational mass**, though this is an **experimental fact** – it does not *have* to the be the same.

3) "*Every **action** must have an **equal** and **opposite** **reaction***".

$$\boxed{\mathbf{F}_{kj} = -\mathbf{F}_{jk}}$$

Where:

- $\mathbf{F}_{ab}$ is the **force** of the a^{th} particle on the b^{th} particle.

Note, however, that the Newton pairs act on **different bodies**.

- **Equations of motion** can be solved by expressing acceleration in one of the following equivalent ways:

$$\boxed{\ddot{\mathbf{x}} = \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}t} = \dot{\mathbf{x}}\frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}\mathbf{x}}}$$

- The <u>**centre of mass**</u> of a **body** or **system of particles** is a **specific point** at which, for **many purposes**, the system's **mass** behaves as if it were **concentrated**. The **position vector** of the centre of mass, **R**, is given by:

$$\boxed{\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\int_{body} \mathbf{r}\,\mathrm{d}m}{\int_{body} \mathrm{d}m} = \frac{1}{M}\int_{body} \mathbf{r}\,\mathrm{d}m}$$

Where:

- o m_i is the <u>**mass**</u> of the particle with <u>**position vector**</u> $\mathbf{r}_i$ from an origin.
- o M is the <u>**total mass**</u> of the body.

Note that in <u>**1 dimension**</u>, $\mathrm{d}m = \rho\,\mathrm{d}l$, where ρ is the <u>**mass per unit length**</u>, and similar results can be derived for 2 and 3 dimensions.

The <u>**principle of superposition**</u> holds, in that to find the **centre of mass** of a **number of bodies**, we can find the centre of mass of each **individual** body and then use that to work out the **common centre of mass**.

Work, Energy, etc…

- We say that <u>**work**</u> is done whenever the **point of application** of a force moves **in the direction** of the force.

For a <u>**displacement**</u> δr at an <u>**angle**</u> θ to the force, the <u>**work done**</u> is given by:

$$\delta W = |\mathbf{F}|\,\delta r\cos\theta$$

If the **displacement** is expressed as a **vector** $\delta\mathbf{r}$, then

$$\boxed{\delta W = \mathbf{F}\cdot\delta\mathbf{r}}$$

For a **large displacement**, we need to **integrate** along the path:

$$\boxed{W = \int_{path} \mathbf{F}\cdot\mathrm{d}\mathbf{r}}$$

- <u>**Power**</u> is simply the <u>**rate of doing work**</u>. If a **small amount** of <u>work</u> δW is done in a **small amount** of <u>time</u> δt, then the <u>**power**</u>, P, is given by

$$P = \frac{\delta W}{\delta t} \quad \text{– and, taking the limit as } \delta t \to 0,\ P = \frac{\mathrm{d}W}{\mathrm{d}t}$$

If the **point of application** of the force is **moving**, it must do **continuous work**, and we can express that power as:

$$\boxed{P = \frac{\delta W}{\delta t} = \frac{\mathbf{F}\cdot\delta\mathbf{r}}{\delta t} \to \mathbf{F}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{F}\cdot\mathbf{v}}$$

- This **energy** can be **dissipated as heat** or **stored inside the system**, for example as **kinetic energy** or **potential energy**.

- The **kinetic energy** of a system is simply given by:

$$\boxed{\text{Kinetic energy} = \sum_i \tfrac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \sum_i \tfrac{1}{2} m_i \left|\mathbf{v}_i\right|^2}$$

  - $\mathbf{v}_i$ is the **velocity** of particle i.
  - m_i is the **mass** of particle i.

  [Prove by considering the work done when $\mathbf{F}$ moves $\delta\mathbf{r}$, express $\mathbf{F}$ as $m\frac{\mathrm{d}\dot{\mathbf{r}}}{\mathrm{d}t}$ and $\delta\mathbf{r}$ as $\dot{\mathbf{r}}\delta t$, and re-arrange].

- The **potential energy** of a system **at a point** is the **energy required** to move a **particle** in that system **from a fixed point** where it is taken to have **potential energy 0** to the **point concerned**:

$$\boxed{\Phi(\mathbf{r}) = \int_{\substack{\text{path from point}\\ \text{of 0 P.E. to } \mathbf{r}}} \mathbf{F}(\mathbf{r})\cdot \mathrm{d}\mathbf{r}}$$

  For a **radial field**, the potential at a distance r from the **centre of force** simplifies to:

$$\boxed{\Phi(r) = \int_a^r \left|\mathbf{F}(r)\right| \, \mathrm{d}r}$$

  Where the **potential energy at $r = a$** is **defined** to be 0.

  A logical consequence of this is that:

$$\boxed{\mathbf{F}(\mathbf{r}) = -\mathbf{grad}(\Phi(\mathbf{r}))}$$

  Specifically, in the case of a **radial field**:

$$\boxed{F(r) = \frac{\mathrm{d}}{\mathrm{d}r}\Phi(r)}$$

- At **maxima** and **minima** of potential energy, $\frac{\mathrm{d}}{\mathrm{d}r}\Phi(r) = \left|\mathbf{F}(r)\right| = 0$. We therefore have an **equilibrium point**.
  - At a **maximum** in $\Phi(r)$, the equilibrium is **unstable**.
  - At a **minimum** in $\Phi(r)$, the equilibrium is **stable**.

Types of Force

- **Weight** acts **vertically downwards** at the centre of gravity, and has **magnitude**
$$|\mathbf{W}| = mg$$
Where:
  - m is the **mass of the body**.
  - g is the **acceleration due to gravity**.
- The **normal reaction force adjusts itself** to ensure that things remain static, and acts **perpendicularly to the surface**. It does **not** necessarily act through the centre of the block, because it is the sum of all action-reaction pairs at the points of contact between the block and the plane's surface. For example, in the case of a block on an inclined plane, the reaction force acts closer to the *lower* part of the block.
- The **friction force** *always* acts **parallel to the surface** and also **adjusts itself** so that the object remains **static**, **up to a certain maximum**, the magnitude of which is given by
$$\left|\mathbf{F}_{static,max}\right| = \mu_{static}\,|\mathbf{N}|$$
Where
  - μ_{static} is the **coefficient of static friction**.
  - $|\mathbf{N}|$ is the **magnitude of the normal reaction force** on the body.

  At that point, **sliding is occurring**, and friction takes on a new, constant value (usually independent on speed) during the motion of the body:
$$\left|\mathbf{F}_{kinetic}\right| = \mu_{kinetic}\,|\mathbf{N}|$$
Where
  - $\mu_{kinetic}$ is the **coefficient of kinetic friction**.
  - $|\mathbf{N}|$ is the **magnitude of the normal reaction force** on the body.
- The tension $\mathbf{F}$ in a spring is given by:
$$\mathbf{F} = kx\hat{\mathbf{x}}$$
Where:
  - x is the **extension of the spring** from its **natural length**.
  - k is the **spring constant** for that spring.
  - $\hat{\mathbf{x}}$ is a **unit vector** pointing **from** the **centre of the spring** to the point under consideration.

Conservation of Linear Momentum

- The **total linear momentum** of an **isolated system** is **constant**. This is a **direct consequence** of Newton's 2^{nd} and 3^{rd}. [Prove by considering a single particle, finding the sum of internal and external forces, and realising that the double sum becomes 0. Then, substitute the expression for the centre of mass.]
- The **impulse of a force** is defined as the **integral** of the force **with respect to time**:

$$\boxed{\mathbf{I} = \int_{\substack{\text{Time of}\\ \text{contact}}} \mathbf{F}\,\mathrm{d}t}$$

  It is a **vector** which represents the **total change** in the **momentum** brought about by the action of the force:

$$\mathbf{I} = \int_{\substack{\text{Time of}\\ \text{contact}}} \mathbf{F}\,\mathrm{d}t = \int_{t_1}^{t_2} \mathbf{F}\,\mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t}\mathrm{d}t = \boldsymbol{\rho}_{t_2} - \boldsymbol{\rho}_{t_1} = \Delta\boldsymbol{\rho}$$

- An **inertial frame** is one in which **Newton's Laws** are **obeyed**. In other words, frames with respect to which "*the motion of an object free of all external forces is motion in a straight line at a constant speed (including 0)*".
- If a given frame is an **inertial frame**, then **any frame** moving at a **constant velocity** with respect to it is **also** an inertial frame. Thus, **accelerating frames** [this includes rotating frames] are ***not*** inertial frames. [This can be proved by noting that the proofs that Newton's laws applies in all inertial frames do not work if the frame is accelerating].
- In fact, the **centrifugal force** is only an **invention** to make **rotating frames appear inertial**. Were that not the case, any object in any inertial frame **rotating** with an object undergoing circular motion would appear to spontaneously move outwards!
- [To derive the **Galilean velocity transformations**, use the fact that $\mathrm{d}t'/\mathrm{d}t = 1$, differentiate the displacement transformation].
- The **zero-Momentum-Frame** is one in which the **total momentum of the system is 0**, and **stays 0** as long as the system is **isolated**. In the **ZMF**, the **kinetic energy** of the system is at its **minimum**.
- The **change in kinetic energy** is the **same** in all **inertial frames**. The **kinetic energy itself**, however, will be **different** since the **speeds** are **different**.
- For **elastic two-body collision** in the ***ZMF***, the **speeds** of the **particles do not change**.

- An **elastic collision** is one in which **KE** is **conserved**. If **KE** is **not conserved**, we have an **inelastic collision**.
- When dealing with **bodies of varying mass**:
  - Work in the **instantaneous rest frame** of the body **before** the **splitting/coalescing**.
  - For calculus to work, the "d" quantities **must** be increases. Now, in the interval δt, the rocket goes from m to $m - \delta m$ and ejects a mass δm. Therefore, since the rocket's mass **increases** by $-\delta m$ in the interval, we say that $\mathrm{d}m = -\delta m$.
  - Ignore second order terms.
  - Use N2 – "**Impulse = Rate of Change of Momentum**".
  - Setting $\mathrm{d}m/\mathrm{d}t$ to the rate of release of fuel is not the way to go! If we do that, we lose the $\mathrm{d}m$ which we need for integration.

Conservation of Angular Momentum

- If an angle of rotation is **small**, it can be described by a **vector**, **perpendicular** to the axis of rotation.
- $\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}$. This is a **polar vector**, pointing along the axis of rotation. This applies to **all** angles, however large.
- Relations, for **circular motion**:
  - $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$
  - $\dot{\mathbf{v}} = -|\boldsymbol{\omega}|^2 \mathbf{r} = -\dfrac{|\mathbf{v}|^2}{|\mathbf{r}|}\hat{\mathbf{r}}$
- The **angular momentum**, denoted **L** is the **moment of the momentum** about a point. So, $\mathbf{L} = \mathbf{r} \times \boldsymbol{\rho} = m\mathbf{r} \times \mathbf{v}$. For a **rigid body**, $\boxed{\mathbf{L} = I\boldsymbol{\omega}}$. We have that

$$\boxed{\dot{\mathbf{L}} = \mathbf{G}_{ext}}$$

  Where $\mathbf{G}_{ext}$ is the **vector sum** of the **external moments** applied to the **system**. Thus, **the total angular momentum of an isolated system is constant**. This applies about **any axis**.
- We define the **angular impulse** as the **integral** of the **moment of a force** with respect to time:

$$\boxed{\mathbf{J} = \int_{\substack{\text{Time of}\\\text{contact}}} \mathbf{r} \times \mathbf{F}\,\mathrm{d}t = \int_{\substack{\text{Time of}\\\text{contact}}} \mathbf{G}\,\mathrm{d}t}$$

We find, as before, that the **angular impulse** is the **change in angular momentum**.

- The **Rotational Version** of **Newton's Second Law** states that

$$\boxed{\mathbf{G} = \dot{\mathbf{L}} = I\dot{\boldsymbol{\omega}}}$$

- The **moment of inertia** of a body is given by $\sum_i m_i \left|\mathbf{r}_i\right|^2$, or, in a continuous form, $\int_{body} \left|\mathbf{r}\right|^2 \, \mathrm{d}m$. Two **laws** pertain to moments of Inertia:

  - The **Parallel Axes Theorem** – If the **moment of inertia** about an axis **through the centre of mass** is I, then the moment of inertia through **another axis**, **parallel to the first** and at a distance a from it is

$$I + Ma^2$$

    (To prove, write the new moment of inertia in terms of the position vector from our new axis *to* the centre of mass, and from there to the point).
  - The **Perpendicular Axes Theorem** – if the **moment of inertia** of a **lamina** about *any two* perpendicular axes **in the plane of the lamina** is I_x and I_y, then the moment of inertia about an axis **mutually perpendicular** to those two is

$$I_x + I_y$$

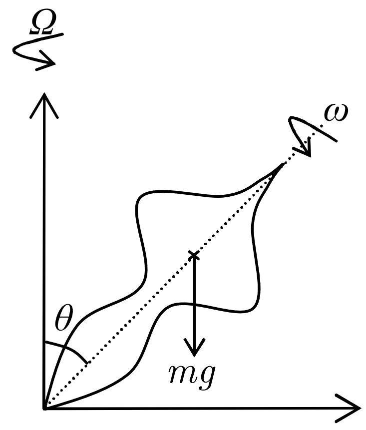
    (Prove by noting that the distance from the z axis squared is equal to the other two distances squared and summed).
- The **rotational kinetic energy** of a body is

$$\frac{1}{2} I \left|\boldsymbol{\omega}\right|^2$$

- The **general motion of a body** can be worked out using the following two facts:
  - The centre of mass of a system moves as if it carries all the mass and is acted upon by the sum of all the external forces.
  - The rotation about an axis through the centre of mass of a body is the result of the sum of the external moments of all the forces acting on that body, and the moment of inertia of the body about that axis.

Gyroscopes

Consider a spinning top, which is spinning about its line of symmetry and precessesing about the z axis:



Now, consider the angular momentum vector from the spinning – if the top has moment of inertia I about that axis, then

$$\mathbf{L} = I\boldsymbol{\omega}$$

Now, the **tip** of this vector $\mathbf{L}$ is rotating in a circle of radius $|\mathbf{L}|\sin\theta$. This means that in a small time interval δt, the momentum changes by a quantity $\boldsymbol{\Omega}\times\mathbf{L}$. Thus,

$$\mathbf{G} = \boldsymbol{\Omega}\times\mathbf{L} = I\boldsymbol{\Omega}\times\boldsymbol{\omega}$$

NOTE: We ignored the momentum $\boldsymbol{\Omega}$ when working out the total angular momentum rotating – thus, our answer is approximate.

Miscellaneous Points

The following might help when working out angles: