Vectors

- Vectors are quantities that have both **magnitude** and **direction**.
- Vector addition is **commutative** and **associative**.
- Multiplication by a scalar is **commutative**, **associative** and **distributive over addition**.
- In general, a basis set must:
    - Contain as many basis vectors as there are dimensions (it must **span** the space).
    - Be such that no basis vector may be described as the sum of others (ie: the basis vectors must be **linearly independent**).

  Any vector may then be expressed as a weighed sum of these basis vectors – the weights are called the **components** of the vector. Most often, we use basis vectors that are **mutually perpendicular**.
- The scalar product of two vectors **a** and **b** is:

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

$$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}).(a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = a_1a_2 + b_1b_2 + c_1c_2$$

  Where θ is the angle between the two vectors.
    - Since $\cos(2\pi - \theta) = \cos\theta$, it matters not whether the inner or outer angler is chosen. Thus, both vectors can either be point **towards each other** or **away from each other**. However, they cannot be pointing opposite directions – if this does happen, the answer comes out negative (because $\cos(\pi - \theta) = -\cos\theta$).
    - The scalar product is **commutative** and **distributive over addition**. This implies that $(\mathbf{a} + \mathbf{b})\cdot(\mathbf{c} + \mathbf{d})$ can be multiplied out normally, since we can just view **a + b** as a single vector. Similarly, $(\lambda\mathbf{a})\cdot\mathbf{b} = \lambda(\mathbf{a}\cdot\mathbf{b})$.
    - $(\mathbf{a} + \mathbf{b})^2 = \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + 2\mathbf{a}\cdot\mathbf{b}$ is simply the **cosine rule**.
    - The scalar product returns a **scalar**. It therefore makes no sense to write things like $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$.
    - Some useful results:
        - $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
        - If $\mathbf{a}\cdot\mathbf{b} = 0$ and neither **a** nor **b** are **0**, then they are **perpendicular**.
        - $\mathbf{a}\cdot\hat{\mathbf{n}}$ ($\hat{\mathbf{n}}$ is a unit vector), is the projection of **a** in the $\hat{\mathbf{n}}$ direction.
- The vector product of two vectors **a** and **b** is:

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}$$

$$(a_1\mathbf{i}+b_1\mathbf{j}+c_1\mathbf{k}).(a_2\mathbf{i}+b_2\mathbf{j}+c_2\mathbf{k})=\left\|\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{pmatrix}\right\|$$

Where θ is the angle between the two vectors, and $\hat{\mathbf{n}}$ is a **unit vector** in the direction which a **right handed screw would move** if turned **from a** to **b**.

- The vector product is **anticommutative** – that is, $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$.
- The vector product is **distributive over addition** – this is, $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$. This, again, implies that $(\mathbf{a}+\mathbf{b})\times(\mathbf{c}+\mathbf{d})$ can be multiplied out normally (we just consider **a + b** as a single vector).
- The vector product of **a** and **b** is perpendicular to both **a** and **b** – in other words, $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=0$.
- The vector product does **not** generalise to n dimensions, because the fact that we can find a unique vector perpendicular to a plane is a special property of 3D space.
- Some useful results:
  - If $\mathbf{a}\times\mathbf{b}=0$ and neither **a** nor **b** are **0**, then they are **parallel**.
  - $|\mathbf{a}\times\mathbf{b}|$ is the **area** of the **parallelogram** with sides **a** and **b**.
  - $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$ is the **area** of the **triangle** with sides **a** and **b**.

- The **scalar triple product** of three vectors **a**, **b** and **c** is given by

$$[\mathbf{a},\mathbf{b},\mathbf{c}]=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

  - **Cyclic permutations** of the three vectors do not change the product. **Anticyclic permutations negate** it.
  - One way to think of the scalar triple product is as the **determinant** of a **matrix** whose **rows** are the three vectors **a**, **b** and **c**.
  - The scalar triple product is a **scalar**, and it is simply the **volume of a parallelepiped** with edges **a**, **b** and **c**.
  - Two useful results, which are a consequence of the fact that this is the area of a parallelepiped:
    - If **any** of the two vectors are **equal**, then the scalar triple product evaluates to 0.
    - The scalar triple product is a good test for coplanarity – **a**, **b** and **c** are coplanar **if and only if** $[\mathbf{a},\mathbf{b},\mathbf{c}]=0$.

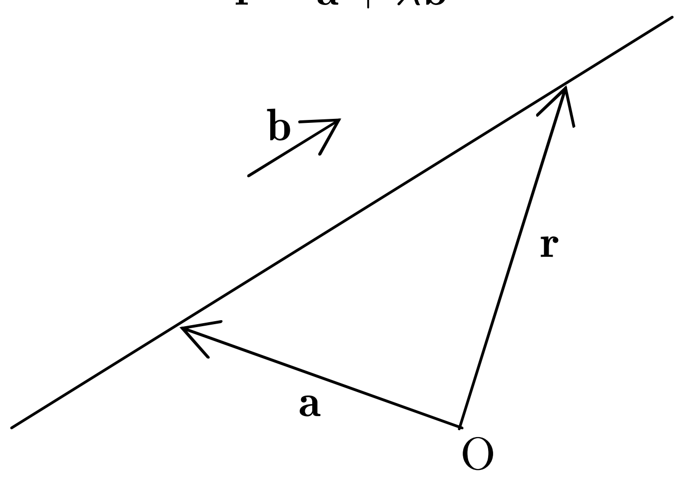
- The **vector triple product** (Arghh!!! ☹) of three vectors **a**, **b** and **c** is given by

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

  - The following can be used to remember this result:

    - $\mathbf{b} \times \mathbf{c}$ is perpendicular to both **b** and **c**, and since $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to that, it must be *in* the plane of **b** and **c**.
    - The bit in the middle (**b**) gets the PLUS sign.
  - The vector triple product isn't associative – ie: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.
  - It can be shown that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.
- The position vector of any point on a line going through point with position vector **a** and parallel to a vector **b** is given by

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$$



Taking components, we find that this can also be expressed as

$$\frac{x - x_1}{\Delta x} = \frac{y - y_1}{\Delta y} = \frac{z - z_1}{\Delta z} = \text{constant}$$

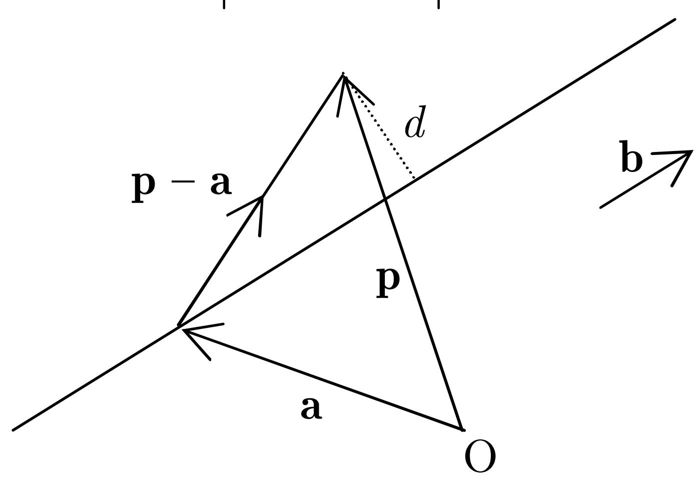
Where (x_1, y_1, z_1) is any point on the line, and $(\Delta x, \Delta y, \Delta z)$ is the difference between any two points on the line.

Taking cross-products on both sides, we get an alternative equation:

$$\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b} \text{ or } (\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$$

(Which should be obvious from the fact that **r** – **a** *must* lie on the line and therefore be parallel to **b**).

  - The minimum distance of a point **p** from a line is calculated by finding the projection of **p** – **a** in the direction perpendicular to the line – this gives the shortest distance as $\left|(\mathbf{p} - \mathbf{a}) \times \hat{\mathbf{b}}\right|$.



  - The minimum distance between two lines $\mathbf{r}_1 = \mathbf{a}_1 + \lambda_1 \mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \lambda_2 \mathbf{b}_2$ is found by projecting any line joining both the lines (for example, $\mathbf{a}_1 - \mathbf{a}_2$) onto the unit common normal of the two lines $(\mathbf{b}_1 \times \mathbf{b}_2)$ – this gives the shortest distance as $\left|(\mathbf{a}_1 - \mathbf{a}_2)\times\widehat{(\mathbf{b}_1 \times \mathbf{b}_2)}\right|$.

- The position vector of any point on a plane containing the points **a**, **b** and **c** is

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

Which can also be written in the form $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{p} + \gamma\mathbf{q}$, where $\alpha + \beta + \gamma = 1$.

Taking a dot product with the vector **normal to the plane**, **n**, we get:

$$\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n} \text{ or } (\mathbf{r} - \mathbf{a})\cdot\mathbf{n}$$

The latter form should be obvious, since **r** – **a** is an arbitrary line in the plane, which is perpendicular to **n**.

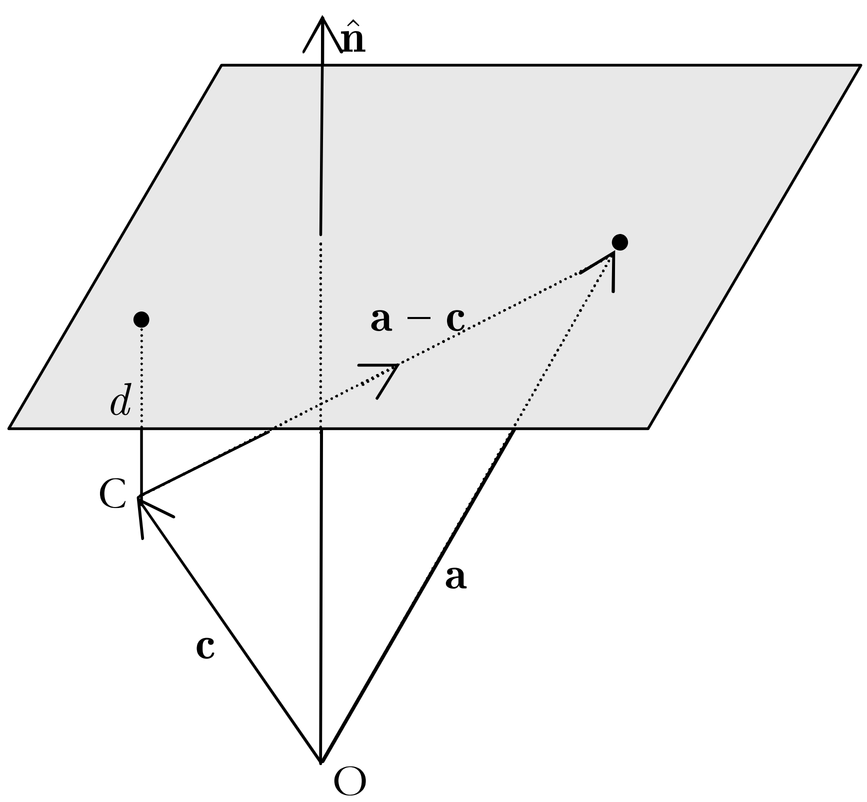
Now, if $\hat{\mathbf{n}}$ is a unit vector in the **n** direction, then $d = \mathbf{a}\cdot\hat{\mathbf{n}}$ is simply the **shortest distance of the plane from the origin**, and we can re-write the equation of the plane as

$$\mathbf{r}\cdot\hat{\mathbf{n}} = d$$

This form is extremely useful in mapping a Cartesian equation of a plane $\alpha x + \beta y + \gamma z = \lambda$ to a vector equation, because the Cartesian equation is basically just

$$\begin{pmatrix} x & y & z \end{pmatrix}\cdot\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} = \lambda$$
$$\mathbf{r}\cdot\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} = \lambda$$

and as long as $\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}$ is nomalised, then λ is the shortest distance from the origin to the plane.



  - The distance between an arbitrary point **c** and a plane can be worked out by projecting any line between the point and the plane (eg: **a** – **c**) in the direction of the **unit** perpendicular. This gives a distance of $(\mathbf{a} - \mathbf{c})\cdot\hat{\mathbf{n}}$.

- If we want the projection of a point on a plane, what we really need is a projection of the point onto a vector perpendicular to $\hat{\mathbf{n}}$. The easiest way to do this is using $|\mathbf{c} \times \hat{\mathbf{n}}|$.
- If we have a line parallel to a plane, and we want to find the distance between the two, we simply need to find any line joining *any* point on the line to *any* point on the plane, and resolve in the $\hat{\mathbf{n}}$ direction.
- To find the line of intersection of two planes, simply cross the two normals – the result will necessarily be parallel to the line of intersection. Furthermore, the point $\mathbf{r} = \alpha\hat{\mathbf{n}}_1 + \beta\hat{\mathbf{n}}_2$ necessarily lies on both planes – use simultaneous equations to find the constants (or, just solve to find a point on both planes).

- A sphere is clearly distinguished by the fact that all points on it are equidistant from a fixed point in space. Thus, the position vector of any point on a sphere centred at $\mathbf{a}$ and with radius r is

$$(\mathbf{r} - \mathbf{a})^2 = r^2$$

- The two sets of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ are called **reciprocal sets** if

$$\mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1$$
$$\text{and } \mathbf{a}' \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{c} = \mathbf{b}' \cdot \mathbf{a} = \mathbf{b}' \cdot \mathbf{c} = \mathbf{c}' \cdot \mathbf{a} = \mathbf{c}' \cdot \mathbf{b} = 0$$

It can be shown that the reciprocal vectors of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are given by:

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \ \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \text{ and } \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

(Note that these only exist if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not coplanar – otherwise, we have division by 0). If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are mutually orthogonal unit vectors, then the two sets of vectors are the same.

We can use this concept to define the components of a vector with respect to basis vectors that are not mutually orthogonal:

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{e}_1')\mathbf{e}_1 + (\mathbf{a} \cdot \mathbf{e}_2')\mathbf{e}_2 + (\mathbf{a} \cdot \mathbf{e}_3')\mathbf{e}_3$$

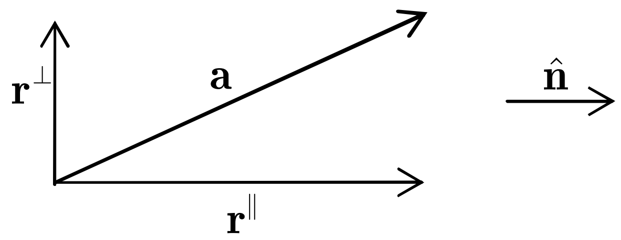
- Vector area
  - We define the **vector area** of a given **plane** surface to have **magnitude** equal to the **area** of the surface, and direction **normal** to the surface. The direction is fixed using extra information, from the right hand screw rule.
  - One way of thinking of vector area is in terms of **flux**. If the surface has vector area $\mathbf{a}$, a flux $\mathbf{F}$ will have a "net flow" of $\mathbf{F} \cdot \mathbf{a}$ through the surface.
  - Thus, the vector area depends only on the **rim** of the surface – not its details. If there is no rim (ie: if we have a **closed surface**), the vector area is $\mathbf{0}$.

- A **component** of the area in a given direction is the **projection of the area** in that direction [again, though, in terms of **net flux** through the surface].

- Random points:
  - If three 3D vectors are linearly dependent, then they **must** be **coplanar**.
  - Remember when solving equations to find where two lines cross that the parameters aren't necessarily the same at the crossing point!
  - To prove that three points are on a line, find a line going from one point to another, and then check the third point does indeed lie on it.
  - The easiest way to prove that the diagonals of a parallelogram intersect is to define an arbitrary parallelogram by its two sides (**a** and **b**) and its diagonal (**c**), and then express the other diagonal in two different ways.
  - When factorising a scalar out of vector equations, it counts twice – thus, $\lambda\mathbf{a}\cdot\lambda\mathbf{b} = \lambda^2(\mathbf{a}\cdot\mathbf{b})$.
  - To find components of a vector **a** parallel and perpendicular to a unit vector $\hat{\mathbf{n}}$ we first note that by the definition of the dot product:

$$\mathbf{r}^{\parallel} = (\mathbf{a}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$$

To find the perpendicular component, we look at the following diagram:



It's clear from this that $\mathbf{r}^{\perp} = \mathbf{a} - \mathbf{r}^{\parallel}$. So:

$$\mathbf{r}^{\perp} = \mathbf{a} - (\mathbf{a}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$$

  - For a vector **a**, we have that

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{i})\mathbf{i} + (\mathbf{a}\cdot\mathbf{j})\mathbf{j} + (\mathbf{a}\cdot\mathbf{k})\mathbf{k} = |\mathbf{a}|\cos\theta_x\mathbf{i} + |\mathbf{a}|\cos\theta_y\mathbf{j} + |\mathbf{a}|\cos\theta_z\mathbf{k}$$

and

$$\hat{\mathbf{a}} = \cos\theta_x\mathbf{i} + \cos\theta_y\mathbf{j} + \cos\theta_z\mathbf{k}$$

This encapsulates *all* direction information, and these are called **direction cosines.**