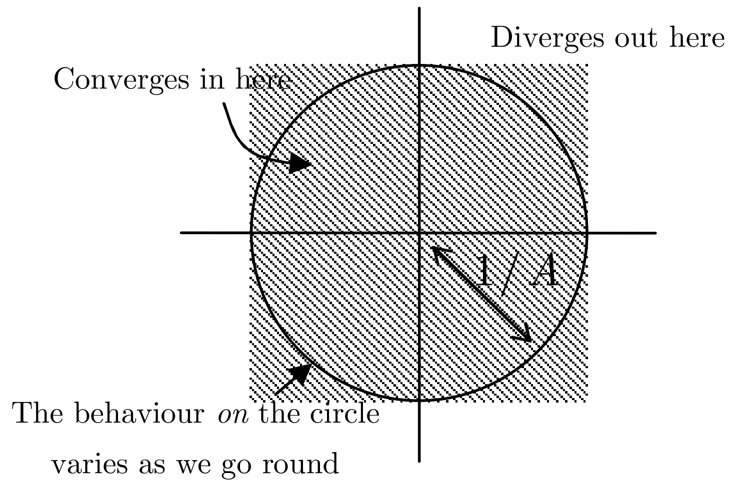
Power Series

- A power series has the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

- The convergence or otherwise of a power series can be determined using the Ratio test, which states that the series converges absolutely (hence the mod sign) if and only if $\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$. Thus, if $A = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, then:
    - The series converges for $|x| < 1/A$.
    - The series diverges for $|x| > 1/A$.
    - At $|x| = 1/A$, the behaviour of the series must be examined using a different test.
    - Note that there is a special case – if A = 0, then the series converges for all x.

  This holds equally well for complex x, and gives rise to a circle of convergence in the complex plane:



  Note that the centre of this circle does not necessarily lie at the origin.

- Power series can be integrated or differentiated term by term. Their interval of convergence then remains the same, but their behaviour at the boundary of that interval does not necessarily remain unchanged.
- A Taylor Series approximates functions about a given point:

> The Taylor Series of f about some point $x = k$ is
> $$f(k+h) = f(k) + hf'(k) + \tfrac{h^2}{2!}f''(k) + \tfrac{h^3}{3!}f'''(k) + \cdots = \sum_{n=0}^{\infty}\frac{h^n}{n!}f^{(n)}(k)$$

  Notes:
    - This is only valid if (a) the derivatives exist and (b) the series is absolutely convergent.

- <u>Taylor's Theorem</u> allows us to estimate the error in a given power series, truncated so that the $\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(k)$ term is the last one, as

$$\frac{h^n}{n!}f^{(n)}(k+\xi)$$

  Where $0 \le \xi \le h$.

  When estimating values using a truncated Taylor Series, the maximum value of this error can be found by differentiating $d\xi$ and setting to 0.

- There are a few standard Taylor series that should be learnt:
  - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ (for all x)
  - $\cos(x) = \text{Re}\left(e^{ix}\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ (for all x)
  - $\sin(x) = \text{Im}\left(e^{ix}\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ (for all x)
  - $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} {}^nC_r x^n$ (for all $|x| < 1$ – the case $|x| = 1$ varies) – note that if n is an integer, this contains only a finite number of terms, because $n - a$ will eventually equal 0.
  - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ (for $-1 < x \le 1$)
- Tips and tricks for finding Taylor series:
  - Finding another, related, series and then integrating to find the one desired can help, but an arbitrary constant should be assumed.
  - To find the Taylor series of, say $\left(f(n)\right)^{-2}$, first expand *f(n)* and then use the Binomial Theorem, collecting terms up to where they are needed.
  - Trigonometric Identities can be used.
  - When expanding about infinity, expand in powers of *1/x* rather than *x*.
- The **Newton-Raphson** method allows us to find better and better solutions to a given equation. Say we have a function *f(x)* and we have a trial solution x_0, then a better estimate to the solution, x_1, is given by:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

