Functions of Several Variables

Introduction

- The **differential form** of a function is

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

- This can be used to work out uncertainties. Say we want the uncertainty in a variable X as a result of uncertainties in variables Y and Z:
  - Consider X as a function of Y and Z
  - Write X in differential form
  - Re-arrange to get an expression for $\mathrm{d}X/X$ in terms of $\mathrm{d}Y/Y$ and $\mathrm{d}Z/Z$ – these are the **percentage errors**.
- The **chain rule** is

$$\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v$$

  It basically applies in a case where f is a function of x and y which are in turn functions of v and u. It can be derived by writing f in differential form, and substituting in x and y in differential form.
- If we have something like $z = f(x, y)$ and we want to find, say $\left(\partial x / \partial y\right)_z$, we simply use the fact that in such a case, $\left(\partial z / \partial x\right)_y = 0$, and then use the chain rule assuming that x is a function of y (and z) – ie, in the form

$$\left(\frac{\partial z}{\partial x}\right)_y = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial x}\right)_z + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z = \left(\frac{\partial z}{\partial x}\right)_y + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z$$

Exact Differentials

- A general differential can be written in the form

$$P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$$

  If there is a function f such that $\mathrm{d}f = P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$, then

$$\left.\begin{aligned} \frac{\partial f}{\partial x} = P(x,y) &\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}P(x,y) \\ \frac{\partial f}{\partial y} = Q(x,y) &\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}Q(x,y) \end{aligned}\right\} \Rightarrow \boxed{\frac{\partial}{\partial y}P(x,y) = \frac{\partial}{\partial x}Q(x,y)}$$

  If this condition is met, then the differential is exact.
- An **integrating factor** is a function which the differential form can be multiplied by to make it exact. Assuming that it is a function of x or y only, it satisfies one of the following two differential equations:

$$\frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x} = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\frac{1}{Q} \qquad \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}y} = -\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\frac{1}{P}$$

Note, however, that integrating factors are not unique!

Maxwell's Relations

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