Chapter 9 – Light-Cone Relativistic Strings

1. Choices for τ

- We previously used the static gauge, in which the world-sheet time is identified with the spacetime coordinate X^0 by $X^0(\tau,\sigma) = c\tau$. We can, however, choose all kinds of different gauges. We choose those in which τ is set equal to a linear combination of the string coordinates:

$$n_\mu X^\mu(\tau,\sigma) = \lambda\tau$$

- To understand what this means, consider two points x_1 and x_2 with the same *fixed* value of τ. We then have

$$n_\mu\left(x_1^\mu - x_2^\mu\right) = 0$$

The vector $\left(x_1^\mu - x_2^\mu\right)$ is clearly on a hyperplane perpendicular to n_μ. If we define the string as the set of points X with constant τ, then we see that **the string with world-sheet time $\boldsymbol{\tau}$ is the intersection of the world-sheet with the hyperplane $\boldsymbol{n \cdot x = \lambda\tau}$**

- We want the interval ΔX^μ between any two points on the string to be spacelike. Now, consider
  - We know that $n_\mu\left(x_1^\mu - x_2^\mu\right) = 0 \Rightarrow n \cdot \Delta x = 0$.
  - If n^μ is timelike, we can anlyse this condition in a frame in which only the time compoenent of n is non-zerio. In that case, Δx clearly has a 0 time compoenent, and is therefore spacelike.

  It turns out that this also works for n^μ null.

- Now – for open strings, p^μ is a conserved quantity. We incorporate this in our Gauge condition and write

$$n \cdot X(\tau,\sigma) = \tilde{\lambda}\left(n \cdot p\right)\tau$$

For open strings attached to D-branes, some components of p^μ are not conserved, but we assume that n is chosen so that $n \cdot p$ is conserved – for this to happen, we need $\boxed{n \cdot \mathcal{P}^\sigma = 0}$ at the string endpoints. Analysing units and working in natural units then gives

$$\boxed{n \cdot X(\tau,\sigma) = 2\alpha'(n \cdot p)\tau} \qquad \text{(open strings)}$$

Not quite sure about the comment that says that gauge isn't Lorentz invariant for all choices of n. I also don't understand how $\boxed{n \cdot \mathcal{P}^\sigma = 0}$ at the endpoints is any requirement – surely we already have $\mathcal{P}^\sigma$ for all endpoints.

2. The Associated σ parameterization for open strings

- In the static gauge, we required constant energy density over the string – in other words, constant $\mathcal{P}^{\tau 0}$. We now require constancy of $n_\mu \mathcal{P}^{\tau\mu} = n \cdot \mathcal{P}^\tau$, as well as $\sigma \in [0, \pi]$.

I don't understand this range condition on sigma?

- From our expression for $\mathcal{P}^{\tau\mu}$, we have

$$\mathcal{P}^{\tau\mu}(\tau,\sigma) = \frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\sigma}\mathcal{P}^{\tau\mu}(\tau,\tilde{\sigma}) \Rightarrow n \cdot \mathcal{P}^{\mu}(\tau,\sigma) = \frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\sigma} n \cdot \mathcal{P}^{\tau}(\tau,\tilde{\sigma})$$

Thus, we can always find a parameterisation in which $n \cdot \mathcal{P}^\tau(\tau,\sigma) = a(\tau)$ (ie: does not depend on σ) by adjusting $\mathrm{d}\tilde{\sigma} / \mathrm{d}\sigma$ accordingly. Further, we note that

$$\int_0^\pi n \cdot \mathcal{P}^\tau \, \mathrm{d}\sigma = n \cdot p = na(\tau)$$

And so

$$\boxed{n \cdot \mathcal{P}^\tau = \frac{n \cdot p}{\pi}} \qquad \text{(open string world-sheet constant)}$$

In this parameterisation, σ for a point is therefore proportional to the amount of $n \cdot p$ momentum carried by the portion of the string between $[0, \sigma]$.

- Now, consider the equations of motion

$$\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0$$

Dotting this with n^μ, we get

$$\frac{\partial}{\partial \tau}\left(n \cdot \mathcal{P}^\tau\right) + \frac{\partial}{\partial \sigma}\left(n \cdot \mathcal{P}^\sigma\right) = 0$$
$$\frac{\partial}{\partial \sigma}\left(n \cdot \mathcal{P}^\sigma\right) = 0$$

Which implies that $n \cdot \mathcal{P}^\sigma$ is independent of σ.

- We have already seen that for open strings, $n \cdot \mathcal{P}^{\sigma} = 0$ at endpoints, which implies that this is the case *everywhere*.

3. The Associated σ parameterization for closed strings

- In this case, we want $\sigma \in \left[0, 2\pi\right]$ and so

$$\boxed{n \cdot \mathcal{P}^{\tau} = \frac{n \cdot p}{2\pi}} \qquad \text{(closed string world-sheet constant)}$$

Because of this factor of two, we write the gauge condition without the factor of two, as

$$n \cdot X = \alpha'\left(n \cdot p\right)\tau$$

I don't understand this range condition on sigma?

- We can still show that $n \cdot \mathcal{P}^{\sigma}$ is independent of σ, but it's now not possible to set it to 0 at any given point. Furthermore, it's unclear *what* point is $\sigma = 0$. We solve this by setting a certain point on a certain string to have these properties. The proof this can be done is in the book.
- There is, however, an obvious ambiguity – our whole parameterisation can be rigidly moved along the string without affecting anything.

4. Summary

- In summary, we have

$$n \cdot \mathcal{P}^{\sigma} = 0$$
$$n \cdot X(\tau, \sigma) = \beta\alpha'\left(n \cdot p\right)\tau$$
$$n \cdot p = \frac{2\pi}{\beta} n \cdot \mathcal{P}^{\tau}$$

Where $\beta = 1$ for closed strings, and $\beta = 2$ for open strings.

- The first condition above, along with an expression for $\mathcal{P}^{\sigma}$ immediately gives us $\boxed{\dot{X} \cdot X' = 0}$. This allows us to simplify our expression for $\mathcal{P}^{\tau\mu}$, and we obtain $\dot{X}^2 + X'^2 = 0$. This is best summarised, together with the first condition above, as

$$\left(\dot{X} \pm X'\right)^2 = 0$$

- We then get the following simplified expressions

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}\left(X^\mu\right)'$$

Feeding into the equations of motion, we get

$$\ddot{X}^\mu - \left(X^\mu\right)'' = 0$$

These are simply wave equations!

- When the string is open, we have the additional requirement that the $\mathcal{P}^{\sigma\mu}$ and therefore the $\left(X^\mu\right)'$ vanish at the endpoints.

4. Solving the wave equation

- Assuming we have a space-filling D-brane and therefore free-boundary conditions at the endpoints, the most general solution to the wave equation is

$$X^\mu(\tau,\sigma) = \frac{1}{2}\left(f^\mu(\tau+\sigma) + g^\mu(\tau-\sigma)\right)$$

Bearing in mind the relation $\mathcal{P}^{\sigma\mu} = -\left(X^\mu\right)' / 2\pi\alpha'$ and the boundary conditions $\mathcal{P}^{\sigma\mu} = 0$, we get

$$\frac{\partial X^\mu}{\partial\sigma} = 0 \qquad \sigma = 0, \pi$$

The boundary condition at 0 informs us that f and g differ at most by a constant, which can be absorbed into f.

$$X^\mu(\tau,\sigma) = \frac{1}{2}\left(f^\mu(\tau+\sigma) + f^\mu(\tau-\sigma)\right)$$

The boundary condition at $\sigma = \pi$ gives

$$\frac{\partial X^\mu}{\partial\sigma}(\tau,\pi) = \frac{1}{2}\left(f^{\mu\prime}(\tau+\pi) - f^{\mu\prime}(\tau-\pi)\right) = 0$$

Since this must be true for all τ, this implies that $f^{\mu\prime}$ is periodic with period 2π.

- We can therefore write

$$f^{\mu\prime}(u) = f_1^\mu + \sum_{n=1}^{\infty}\left(a_n^\mu \cos nu + b_n^\mu \sin nu\right)$$

$$f^\mu(u) = f_0^\mu + f_1^\mu u + \sum_{n=1}^{\infty}\left(A_n^\mu \cos nu + B_n^\mu \sin nu\right)$$

Substituting and simplifying, we get

$$X^\mu(\tau,\sigma) = f_0^\mu + f_1^\mu \tau + \sum_{n=1}^{\infty}\left(A_n^\mu \cos n\tau + B_n^\mu \sin n\tau\right)\cos n\sigma$$

We write

$$\begin{aligned} A_n^\mu \cos n\tau + B_n^\mu \sin n\tau &= -\frac{i}{2}\left(\left(B_n^\mu + iA_n^\mu\right)e^{in\tau} - \left(B_n^\mu - iA_n^\mu\right)e^{-in\tau}\right) \\ &= -i\sqrt{\frac{2\alpha'}{n}}\left(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau}\right) \end{aligned}$$

f_1 can be shown to be proportional to the momentum carried by the string (by integrating the momentum density), and we can say $f_0^\mu = x_0^\mu$. We then get

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau - i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau}\right)\frac{\cos n\sigma}{\sqrt{n}}$$

Clearly, this corresponds to the zero-mode of the string, its momentum, and its oscillations.

I don't understand how we can just declare $f_0^\mu = x_0^\mu$

- Now, let's define lots of notation

$$\alpha_0^\mu = p^\mu \sqrt{2\alpha'}$$

$$\alpha_n^\mu = a_n^\mu \sqrt{n} \qquad \alpha_{-n}^\mu = \left(\alpha_n^\mu\right)^* = a_n^{\mu *}\sqrt{n}$$

- We can then write

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma$$

And we then have

$$\dot{X}^\mu = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in\tau}\cos n\sigma \qquad X^{\mu\prime} = -i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in\tau}\sin n\sigma$$

And

$$\dot{X}^\mu \pm X^{\mu\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} a_n^\mu e^{-in\left(\tau\pm\sigma\right)}$$

We need to make sure that this satisfies the boundary conditions.

5. Light-cone solutions of equations of motion

- To move into a light-cone gauge, we trade coordinates x^0 and x^1 for coordinates x^+ and x^-, and we set a gauge that has $n \cdot X = X^+$. This gives us, via the relations defined above,

$$X^+(\tau,\sigma) = \beta\alpha' p^+ \tau \qquad\qquad p^+ = \frac{2\pi}{\beta}\mathcal{P}^{\tau +}$$

  The second equation tells us that p^+ density is constant along the string.

- We want to try and show that all the dynamics are in the transverse coordinate X^I (ie: not including x^+ and x^-). First, consider the constraint equation, using the dot product in light-cone coordinates

$$-2\left(\dot{X}^+ \pm X^{+\prime}\right)\left(\dot{X}^- \pm X^{-\prime}\right) + \left(\dot{X}^I \pm X^{I\prime}\right)^2 = 0$$

  From the equation above for X^+, we have that $X^{+\prime} = 0, \dot{X}^+ = \beta\alpha' p^+$, and so

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{2\beta\alpha' p^+}\left(\dot{X}^I \pm X^{I\prime}\right)^2$$

  We have assuming that $p^+ > 0$. This only fails when $p^+ = 0$, which only occurs for a massless particle travelling exactly in the negative x^1 direction. This is an unusual occurrence, but when it does occur, the light-cone formalism will not apply.

- These define $\dot{X}^-$ and $X^{-\prime}$ in terms of the X^I, and therefore completely determine X^- up to an integration constant. All we need is the value of X^- at some point on the world sheet, and integrate $\mathrm{d}X^- = \dot{X}^- \,\mathrm{d}\tau + X^{-\prime}\,\mathrm{d}\sigma$. On a closed string, we further have a condition that $\int_0^{2\pi} X^{-\prime}\,\mathrm{d}\sigma = 0$, to ensure that the contour we choose around the string does not affect the value of X^-. Thus, the string motion is characterised by X^I, p^+ and x_0^-, where the last item is the constant of integration.

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Chapter 10 – Light Cone Fields & Particles

1. Action for Scalar fields

- A scalar field is a single real function of spacetime; $\phi(\boldsymbol{x}, t) \equiv \phi(x)$.
- A natural choice for the **Lagrangian Density** and **action** of a field that treats **time** and **space** on an **equal footing** is

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \qquad\qquad S = \int \mathcal{L}\,\mathrm{d}^D x$$

  Where $D = d + 1$ is the total number of dimensions.
- This is for a **free scalar field with mass m** (a **free field** is one in which the equation of motion is **linear** in the field, which require the Lagrangian to be **quadratic** in the field).

  Why are those densities and not the Hamiltonian itself?
- The **momentum conjugate** to the **field** is given by

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_0\phi\right)} = \partial_0\phi$$

  And the Hamiltonian density by

$$\mathcal{H} = \Pi\left(\partial_0\phi\right) - \mathcal{L} = \frac{1}{2}\left(\Pi^2 + \left(\nabla\phi\right)^2 + m^2\phi^2\right)$$

  How does those refer to T, V' and V, and why did we expect that?

  The energy is then given by the Hamiltonian

$$E = H = \int \mathcal{H}\,\mathrm{d}^d x$$

  Where d is the number of **space** dimensions.

  Why use the *space* dimensions here?

2. Equation of motion and classical solutions

- Varying the action, we get an equation of motion

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = 0$$
$$\left(\partial^2 - m^2\right)\phi = 0$$
$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi - m^2\phi = 0$$

This is the **Klein-Gordon equation**.

- Now, finding plane-wave solutions to the classical scalar field. Consider (note the two terms, to ensure the solution is real)

$$\phi(t,\boldsymbol{x}) = a e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - Et)} + a^* e^{i(-\boldsymbol{p}\cdot\boldsymbol{x} + Et)}$$

Where $\boldsymbol{p}$ is an arbitrary vector, and the form of the differential equation requires

$$E = \pm E_p \qquad\qquad E_p = \sqrt{\boldsymbol{p}^2 + m^2}$$

A general solution can be obtained by superimposing all the possible solutions above (note that $\boldsymbol{p}$ is continuous, so we get an integral). However, it has no simple QM interpretation, because the second term represents a particle of *negative* energy…

- To analyse the scalar field equation, it helps to work in Fourier space

$$\phi(x) = \int \frac{1}{(2\pi)^D} e^{ip\cdot x} \phi(p)\,\mathrm{d}^D p$$

Note that we require $[\phi(x)] = [\phi(x)]^* \Rightarrow [\phi(p)] = [\phi(-p)]^*$

- Substituting the expression for $\phi(x)$ into the equation of motion, we find that the field **must be 0 unless it lies on a "mass shell"**, on which $p^2 + m^2 = 0 \Rightarrow E^2 = \boldsymbol{p}^2 + m^2$. This is a **hyperboloid**, described by the set of points $(\pm E_p, \boldsymbol{p})$ for all $\boldsymbol{p}$.

Why do virtual particles not lie on the mass shell?

- We note that any point p^μ on the mass shell has a **single number** associated with it, because the complex number has *two degrees of freedom*, and the condition $[\phi(p)] = [\phi(-p)]^*$ takes away one of them. We thus is there is **one classical degree of freedom per point on the mass shell**.
- Don't understand page 200

3. Scalar Quantum Field Theory

- When we move to quantum mechanics, the **dynamical variables** turn to **operators**. Thus, our **field** becomes a **field operator** (and we also have

momentum and **energy operators**). The state space is described using a set of **particle states**.

- Let's write the plane-wave solutions to the KG equations above in more general form, and with normalisation factors:

$$\phi_p(t,\boldsymbol{x}) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2E_p}}\left(a(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^*(t)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)$$

Why do we add these particular pre-factors?

We can imagine this as a field in a box of sides L_i, with periodicity

$$p_i L_i = 2\pi n_i$$

We evaluate the scalar field action and Hamiltonian for this field – we'll need to square the field, square its time derivative and square its gradient. All terms with spatial dependence will integrate to 0, and the others will cancel the V terms in the field. We then get

$$S = \int\left(\frac{1}{2E_p}\dot{a}^*(t)\dot{a}(t) - \frac{1}{2}E_p a^*(t)a(t)\right)\mathrm{d}t$$

$$H = \frac{1}{2E_p}\dot{a}^*(t)\dot{a}(t) - \frac{1}{2}E_p a^*(t)a(t)$$

If we write $a(t) = q_1(t) + iq_2(t)$, the action becomes

$$S = \sum_{i=1}^{2}\int\left(\frac{1}{2E_p}\dot{q}_i^2(t) - \frac{1}{2}E_p q_i^2(t)\right)\mathrm{d}t$$

This is the action for two harmonic oscillators, with associated momenta

$$p_i(t) = \frac{\partial L}{\partial \dot{q}_i} = \frac{\dot{q}_i(t)}{E_p} \Rightarrow p_1 + ip_2 = \frac{1}{E_p}\dot{a}$$

And equations of motion

$$\ddot{q}_i(t) = -E_p^2 q_i(t) \Rightarrow \ddot{a}(t) = -E_p^2 a(t)$$

With solutions

$$\boxed{a(t) = a_p e^{-iE_p t} + a_{-p}^* e^{iE_p t}}$$

Feeding this into the Hamiltonian, we find

$$H = E_p\left(a_p^* a_p + a_{-p}^* a_{-p}\right)$$

- We postulate, and can check, that the a_p and a_{-p} are annihilation operators, with q_1 and q_2 being the relevant coordinates. We then have

$$\left[a_p, a_p^\dagger\right] = 1 \qquad \left[a_{-p}, a_{-p}^\dagger\right] = 1$$

With a Hamiltonian and momentum

$$H = E_p\left(a_p^\dagger a_p + a_{-p}^\dagger a_{-p}\right) \qquad \boldsymbol{P} = \boldsymbol{p}\left(a_p^\dagger a_p - a_{-p}^\dagger a_{-p}\right)$$

- More generally, including *all* momenta, we get

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{k}}^\dagger\right] = \delta_{\boldsymbol{p},\boldsymbol{k}} \qquad \left[a_{\boldsymbol{p}}, a_{\boldsymbol{k}}\right] = \left[a_{\boldsymbol{p}}^\dagger, a_{\boldsymbol{k}}^\dagger\right] = 0$$

And

$$H = \sum_{\boldsymbol{p}} E_{\boldsymbol{p}} a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} \qquad \boldsymbol{P} = \sum_{\boldsymbol{p}} \boldsymbol{p} a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}$$

The field operator, with contributions from all momenta, is then

$$\phi(t, \boldsymbol{x}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{p}} \frac{1}{\sqrt{2E_p}}\left(a_p e^{i\left(-E_p t + i\boldsymbol{p}\cdot\boldsymbol{x}\right)} + a_p^\dagger e^{i\left(E_p t - \boldsymbol{p}\cdot\boldsymbol{x}\right)}\right)$$

- We then define $\left|\Omega\right\rangle$ as a **vacuum state**, containing *no* particles, and for which $a_p\left|\Omega\right\rangle = 0$. A state containing particles with momenta $p_1 \ldots p_k$ is then

$$\left|\psi\right\rangle = a_{p_1}^\dagger a_{p_2}^\dagger \cdots a_{p_k}^\dagger \left|\Omega\right\rangle$$

With

$$P\left|\psi\right\rangle = \sum_k p_k \qquad H\left|\psi\right\rangle = \sum_k E_{p_k}$$

The **number operator**, N, gives the number of particles in the state

$$N = \sum_{\boldsymbol{p}} a_p^\dagger a_p$$

To prove the above, consider that

$$a_p^\dagger a_p a_p^\dagger \left|\Omega\right\rangle = \left(a_p^\dagger a_p a_p^\dagger - 0\right)\left|\Omega\right\rangle = \left(a_p^\dagger a_p a_p^\dagger - a_p^\dagger a_p^\dagger a_p\right)\left|\Omega\right\rangle = a_p^\dagger\left[a_p, a_p^\dagger\right]\left|\Omega\right\rangle$$

- At the quantum level, we focus on the one-particle states, which lie on the physical part of the mass shell, with positive energy, $p^0 = E > 0$. We thus have a single particle state for each point on the physical mass shell, labelled by its momentum $\boldsymbol{p}$.
- In light-cone coordinates, the energy is p^- and the momenta are characterised by p^T and p^+. Thus, we label the oscillators with p^T and p^+

$$\text{One particle state} = a_{p^+,p^T}^\dagger \left|\Omega\right\rangle$$

$$\hat{p}^? = \sum_{p^+,p^T} p^? a_{p^+,p^T}^\dagger a_{p^+,p^T} \qquad \hat{p}^- = \sum_{p^+,p^T} \frac{1}{2p^+}\left(p^I p^I + m^2\right) a_{p^+,p^T}^\dagger a_{p^+,p^T}$$

Where, in the last operator, we have used the fact that

$$p^2 + m^2 = 0 \Rightarrow p^- = \left(p^I p^I + m^2\right) / 2p^+$$

4. Maxwell Fields & Photon States

- In the case of Maxwell fields, we have Gauge Invariance, in which $\partial_\nu A_\mu$ is invariant under the transformation $\delta A_\mu = \partial_\mu \varepsilon$. This yields field equations of the form [see previous chapter]

$$\partial_\nu F^{\mu\nu} = 0$$
$$\partial_\nu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0$$
$$\boxed{\partial^2 A^\mu - \partial^\mu \left(\partial \cdot A\right) = 0}$$

How do we get the last step?

Compared to the equation for a scalar field, this is conspicuous in its absence of a mass term.

- Transferring this to momentum space, we get

$$A^\mu(x) = \int \frac{\mathrm{d}^D p}{\left(2\pi\right)^D} e^{ipx} A^\mu(p) \qquad\qquad A^\mu(-p) = \left[A^\mu(p)\right]^*$$

Substituting into the field equations, we get

$$p^2 A^\mu - p^\mu \left(p \cdot A\right) = 0$$

How do we get this?

- We can also Fourier Transform the gauge transformation

$$\delta A_\mu(p) = ip_\mu \varepsilon(p)$$
$$\delta A^+ = ip^+\varepsilon \qquad \delta A^- = ip^-\varepsilon \qquad \delta A^I = ip^I\varepsilon$$

With $\varepsilon(-p) = \varepsilon^*(p)$.

- We then fix our gauge as follows – we set $A^{+\prime} = A^+ + ip^+\varepsilon$, and $\varepsilon = iA^+ / p^+$, which the gives us

$$\boxed{A^+(p) = 0}$$

This fixes the gauge, because any addition transformations make A^+ non-zero (with the exception of $\varepsilon(p) = \varepsilon(p^-, p^I)\delta(p^+)$, because $p^+\varepsilon = 0$)

How does the exception work?

- This Gauge condition simplifies the equation of motion. First, take $\mu = 0$, and get

$$p^+\left(p \cdot A\right) = 0 \Rightarrow p \cdot A = 0$$
$$\Rightarrow -p^+A^- - \cancel{p^-A^+} + p^I A^I = 0$$
$$A^- = \frac{p^I A^I}{p^+}$$

And all that remains from the field equation is

$$p^2 A^\mu(p) = 0$$

This is automatically satisfied for $\mu = +$. For $\mu = I$, this leads to a set of conditions, and for $\mu = -$, it is satisfied because of these conditions and the formula for A^- above.

- Each of the equations for $\mu = I$ correspond to the equations of motion for a massless scalar. Thus
  - When $p \neq 0$ [ie: a massive particle], the full Gauge field vanishes.
  - When $p = 0$, each of the A^I are independent, and A^- is determined by the relation above.

  We therefore have $D - 2$ degrees of freedom per point on the mass shell.

- *Note*: we can show that there are no degrees of freedom for $p^2 \neq 0$ by noting that if a field only differs from the 0 field by a Gauge Transformation $\partial_\mu \chi$, then it is effectively equivalent to the 0 vector. We call the field *pure Gauge*. In momentum space, we need

$$\text{pure gauge}: \quad A_\mu(p) = ip_\mu \chi(p)$$

If we can show that our field has this form for $p^2 \neq 0$, then we can show that it effectively vanishes for $p^2 \neq 0$. Taking the equation of motion

$$p^2 A_\mu = p_\mu \left(p \cdot A\right)$$

And using the fact that $p^2 \neq 0$, we can divide by p^2

$$A_\mu = ip_\mu \frac{-ip \cdot A}{p^2}$$

This is precisely in the pure Gauge form.

- Finally, let's consider photon states. We can introduce oscillators for each of the A^I fields; namely, $a^I_{p^+,p^T}$ and $a^{I\dagger}_{p^+,p^T}$. Each of the I represent a different possible polarisation – there are $D - 2$ of each of these independent states for each point on the mass shell. A general one-photon state with p^+ and p^T contains a linear superposition of these polarisations:

$$\left|\psi\right\rangle = \sum_{I=2}^{D-1} \xi_I a^{I\dagger}_{p^+,p^T} \left|\Omega\right\rangle$$

Where the vector ξ dictates how we superpose each of the polarisations.

- For $D = 4$, we get $D - 2 = 2$; the familiar two polarisations of light.

5. Gravitational Fields and Graviton States

- In GR, the dynamical field variable is the metric $g_{\mu\nu}(x)$, which, in weak fields, can be taken to be $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$, both g and h being symmetric under exchange of indices, and with

$$\partial^2 h^{\mu\nu} - \partial_\alpha \left(\partial^\mu h^{\nu\alpha} + \partial^\nu h^{\mu\alpha}\right) + \partial^\mu \partial^\nu h = 0$$

The momentum-space version is (if there were sources, there'd be an extra term including the energy-momentum tensor for these)

$$S^{\mu\nu}(p) \equiv p^2 h^{\mu\nu} - p_\alpha \left(p^\mu h^{\nu\alpha} + p^\nu h^{\mu\alpha}\right) + p^\mu p^\nu h = 0$$

Where $h = \eta^{\mu\nu} h_{\mu\nu} = h^\mu_\mu$.

- The equation of motion are invariant under the Gauge transformations

$$\delta_0 h^{\mu\nu}(p) = ip^\mu \varepsilon^\nu(p) + ip^\nu \varepsilon^\mu(p)$$

Where the gauge parameter is a vector, and gauge invariance is effectively reparameterisation invariance. To see how, first compute

$$\delta_0 h = \eta_{\mu\nu} \delta_0 h^{\mu\nu} = i\eta_{\mu\nu} \left(p^\mu \varepsilon^\nu + p^\nu \varepsilon^\mu\right) = 2ip \cdot \varepsilon$$

And then see that $\delta S^{\mu\nu}$ does indeed vanish.

- Since the metric is symmetric and has two indices (+, – or I) we must consider

$$\left(h^{IJ}, h^{+I}, h^{-I}, h^{+-}, h^{++}, h^{--}\right)$$

By writing the gave conditions for all the above that include a +, it turns out we can set all these objects to 0. The Gauge conditions then become

$$\boxed{h^{++} = h^{+-} = h^{+I} = 0}$$

- Setting $\mu = \nu = +$ in the equations of motion, we find that

$$\left(p^+\right)^2 h = 0 \Rightarrow h = 0$$
$$\Rightarrow -2h^{+-} + h^{II} = 0$$
$$\Rightarrow \boxed{h^{II} = 0}$$

The equation of motion reduces to

$$p^2 h^{\mu\nu} - p^\mu p_\alpha h^{\nu\alpha} - p^\nu p_\alpha h^{\mu\alpha} = 0$$

Setting $\mu = +$, we get $p^+\left(p_\alpha h^{\nu\alpha}\right) = 0 \Rightarrow p_\alpha h^{\nu\alpha} = 0$. And so the equation of motion reduces to

$$\boxed{p^2 h^{\mu\nu} = 0}$$

Furthermore, from $p_\alpha h^{\nu\alpha} = 0$, we can find an equation for the h with a – index in terms of the transverse h^{IJ}.

- For any field with a + index, the equation of motion holds trivially. For any field with a – index, it also holds because we found these fields in terms of the transverse h^{IJ}. For the remaining transverse components
  - $h^{IJ}(p) = 0$ for $p^2 \neq 0$ [massive particles]
  - $h^{IJ}(p)$ is unconstrained for $p^2 = 0$, except for requiring $h_{II}(p) = 0$.
- Thus, the degrees of freedom are carried by a *symmetric*, *traceless*, *tranverse* tensor field h^{IJ}, the components of which satisfy the equations of motion of a massless scalar. This has as many compoenents as a symmetric, traceless square matrix of size $D – 2$. Namely

$$n(D) = \frac{1}{2}D\left(D-3\right)$$

The one-graviton states of momentum $\left(p^+, \boldsymbol{p}^T\right)$ are then

$$\left|\psi\right\rangle = \sum_{I,J=2}^{D-1} \xi_{IJ} a^{IJ\dagger}_{p^+,p^T}\left|\Omega\right\rangle \qquad \xi_{II} = 0$$

Chapter 11 – The Relativistic Quantum Point Particle

1. The Light-Cone Point Particle

- Thinking of τ as a time variable, and the $x^\mu(\tau)$ as coordinates, we define an action and a Lagrangian as follows:

$$S = \int_{\tau_i}^{\tau_f} L \,\mathrm{d}\tau \qquad L = -m\sqrt{-\dot{x}^2} \qquad \dot{x}^2 = \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

- The momentum is then given by

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}$$

Which clearly satisfies

$$p^2 + m^2 = 0$$

- The Euler-Lagrange equations give

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = 0$$

- We define a light-cone Gauge for the particle as follows

$$x^+ = \frac{1}{m^2}p^+\tau$$

- Now, consider the + component of momentum

$$p^+ = \frac{m}{\sqrt{-\dot{x}^2}}\dot{x}^+ = \frac{1}{\sqrt{-\dot{x}^2}}\frac{p^+}{m}$$

$$\dot{x}^2 = -\frac{1}{m^2}$$

We can now simplify the expression for momentum

$$p_\mu = m^2\dot{x}_\mu$$

And the equation of motion gives

$$\ddot{x}_\mu = 0$$

- Expanding the $p^2 + m^2 = 0$ in light-cone components, we can obtain

$$\boxed{p^- = \frac{1}{2p^+}\left(p^I p^I + m^2\right)}$$

- From the expression for momentum, we obtain

$$\frac{\mathrm{d}x^?}{\mathrm{d}\tau} = \frac{1}{m^2}p^? \Rightarrow x^?(\tau) = x_0^? + \frac{p^-}{m^2}\tau$$

The Gauge condition tells us that $x_0^+ = 0$.

- Our independent dynamical variables are therefore

$$\left(x^I, x_0^-, p^I, p^+\right)$$

2. Quantising the Point Particle

- Before we quantise the point particle, we need to decide what operators we will use to describe the motion. It seems that the dynamical variables form excellent choices, with

$$\left[x^I, p^J\right] = i\eta^{IJ} = i\delta_{IJ} \qquad \left[x_0^-, p^+\right] = i\eta^{-+} = -i$$

- The operators x^+, x^- and p^- can be defined using those operators, using the relations we've already defined above.
- Since p^- is the light-cone energy, we expect it to generate x^+ evolution: $\partial / \partial x^+ \Leftrightarrow p^-$ (since x^+ is the time component). The Hamiltonian, however, generates τ evolution. But since $x^+ = p^+\tau / m^2$, we can anticipate that

$$\frac{\partial}{\partial \tau} = \frac{p^+}{m^2}\frac{\partial}{\partial x^+} \Leftrightarrow \frac{p^+}{m^2} p^-$$

And so we postulate that

$$H(\tau) = \frac{p^+(\tau)}{m^2} p^-(\tau) = \frac{1}{2m^2}\left(p^I(\tau)p^I(\tau) + m^2\right)$$

- Now, we know that $i\dot{\xi} = \left[\xi, H\right]$, and so from this, we can deduce that
  - $\dot{p}^+(\tau) = \dot{p}^I(\tau) = 0$ – this is good, since these are constants of the motion. We can therefore write $p^+(\tau) = p^+$ and $p^I(\tau) = p^I$.
  - $\dot{x}^I(\tau) = p^I / m^2$. This is, one again, in accord with our classical expectations, and allows us to write $x^I(\tau) = x_0^I + p^I\tau / m^2$.
  - We do indeed get $\dot{x}_0^- = 0$; expected, since it's a constant of integration.
  - $p^-(\tau)$ is a function of the p^I only, and is therefore clearly constant.
  - x^+ and x^- both have explicit time dependence, and we can find that $\dot{x}^-(\tau) = p^- / m^2$ and that $\dot{x}^+(\tau) = p^+ / m^2$. Both of which are as expect classically.

So it looks like our choice of H is good!

- To complete our description, we need to find an appropriate state space. In our CSCO, we can only choose one operator from each of the pair (x^-, p^+) and (x^I, p^I). Because momentum space is usually convenient, we write the states of the quantum point particle as $\left|p^+, \boldsymbol{p}_T\right\rangle$.
- The operators then all act on these states as one might expect – most importantly

$$H\left|p^+, \boldsymbol{p}_T\right\rangle = \frac{1}{2m^2}\left(p^I p^I + m^2\right)\left|p^+, \boldsymbol{p}_T\right\rangle$$

  From which it then follows that the time-dependent states

$$\exp\left[-i\frac{1}{2m^2}\left(p^I p^I + m^2\right)\tau\right]\left|p^+, \boldsymbol{p}_T\right\rangle$$

  Satisfy the Schrodinger Equation.
- More generally, consider the time-dependent superpositions of the basis states

$$\left|\Psi, \tau\right\rangle = \int \mathrm{d}p^+ \mathrm{d}\boldsymbol{p}_T \psi(\tau, p^+, \boldsymbol{p}_T)\left|p^+, \boldsymbol{p}_T\right\rangle$$

  And we see that ψ is none other than the momentum-space wavefunction:

$$\left\langle p^+, \boldsymbol{p}_T \middle| \Psi, \tau\right\rangle = \psi(\tau, p^+, \boldsymbol{p}_T)$$

  Taking the Schrodinger Equation for state $\left|\Psi, \tau\right\rangle$ – namely $i\frac{\partial}{\partial\tau}\left|\Psi, \tau\right\rangle = H\left|\Psi, \tau\right\rangle$ and feeding in the general superposition above, we recover a Schrodinger equation for ψ.

3. Quantum particle and scalar particles

- There is a natural identification of the quantum states of a relativistic point particle of mass m with one-particle states of the quantum theory of a scalar field of mass m

$$\left|p^+, \boldsymbol{p}_T\right\rangle \leftrightarrow a^\dagger_{p^+, \boldsymbol{p}_T}\left|\Omega\right\rangle$$

  We might have expected this correspondence by noticing that the scalar field equations, in light-come coordinates, looks identical to the Schrodinger equation in light-cone coordinates.

- The scalar field theory looks more complete, though, because it allows multi-particle states. What has in fact happened is that we have gone through two levels of quantisation.
  - *First quantisation* involves substituting each of the classical coordinates for quantum operators and obtaining a Schrodinger equation for the wavefunction (a **field**).
  - *Second quantisation* involves quantising the field that we found in first quantisation, and obtaining multi-particle states.

3. Light-cone momentum operators

- Since the Lagrangian depends only on derivatives, it is invariant under the translations $\delta x^\mu(\tau) = \varepsilon^\mu$, where ε^μ is a constant. The resulting conserved charges are momenta, and, in the quantum theory, they generate the symmetry transformation via commutation.
- If we had carried out Lorentz-invariant quantisation of the point particle, the operators we would have used would have been the $x^\mu(\tau)$ and $p^\mu(\tau)$. In that case, the commutation relations would have been
$$\left[x^\mu(\tau), p^\nu(\tau)\right] = i\eta^{\mu\nu} \qquad \left[x^\mu, x^\nu\right] = \left[p^\mu, p^\nu\right] = 0$$
Now, we'd like to check that $i\varepsilon_\rho p^\rho(\tau)$ does indeed generate the symmetry transformation via commutation:
$$\left[i\varepsilon_\rho p^\rho(\tau), x^\mu(\tau)\right] = i\varepsilon_\rho\left(-i\eta^{\rho\mu}\right) = \varepsilon^\mu = \delta x^\mu(\tau)$$
- However, it's clear that the above commutators don't work in the light-cone gauge. They predict that $\left[x^+(\tau), p^-(\tau)\right] = -i$, whereas we predicted that they were equal to 0.
- That said, let us try and expand the generator $i\varepsilon_\rho p^\rho(\tau)$ in light-cone coordinates [note that the momenta are τ–independent]
$$i\varepsilon_\rho p^\rho(\tau) = -i\varepsilon^- p^+ - i\varepsilon^+ p^- + i\varepsilon^I p^I$$
Let's test it in a number of cases
  - $\boxed{\varepsilon^I \neq 0, \varepsilon^\pm = 0}$, in which case we have
$$\left[i\varepsilon_\rho p^\rho(\tau), x^\mu(\tau)\right] = i\varepsilon^I\left[p^I, x^\mu(\tau)\right]$$

Taking this for $\mu = J,+,-$ gives exactly the results we would expect. Only a δx^J compoenent.

- o $\boxed{\varepsilon^- \neq 0, \varepsilon^+ = \varepsilon^I = 0}$, similar sensible results are obtained.
- o $\boxed{\varepsilon^+ \neq 0, \varepsilon^- = \varepsilon^I = 0}$ is more complicated, because p^- is a nontrivial function of other momenta. We then have

$$\delta x^\mu(\tau) = \left[i\varepsilon_\rho p^\rho(\tau), x^\mu(\tau)\right] = -i\varepsilon^+\left[p^-, x^\mu(\tau)\right]$$

Which doesn't satisfy our naïve expectations. In fact, we find that

$$\delta x^+(\tau) = -i\varepsilon^+\left[p^-, x^+(\tau)\right] = -i\varepsilon^+\frac{\tau}{m^2}\left[p^-, p^+\right] = 0$$

$$\delta x^I(\tau) = -i\varepsilon^+\left[p^-, x^I(\tau)\right] = -i\varepsilon^+\frac{1}{2p^+}\left[p^I p^I, x^I(\tau)\right] = -\varepsilon^+\frac{p^I}{p^+}$$

$$\delta x^-(\tau) = -i\varepsilon^+\left[p^-, x^-\right] = -i\varepsilon^+\left[p^-, x_0^- + \frac{p^-}{m^2}\tau\right] = -i\varepsilon^+\left[p^-, x_0^-\right] = -\varepsilon^+\frac{p^-}{p^+}$$

(To find the last commutator, we note that p^- depends on p^+, and that what we actually need to find is

$$\left[x_0^-, \left(p^+\right)^{-1}\right] = \left\{\left(p^+\right)^{-1} p^+\right\} x_0^-\left(p^+\right)^{-1} - \left(p^+\right)^{-1} x_0^-\left\{p^+\left(p^+\right)^{-1}\right\}$$

which gives the result above)

- We need to understand the translations p^- generates. It turns out we can understand them as a translation $\delta x^\mu = \varepsilon^\mu$ as well as a reparameterisation. The general form of a reparameterisation involves $\tau \to \tau' = \tau + \lambda(\tau)$. In other

$$x^\mu(\tau) \to x^\mu\left(\tau + \lambda(\tau)\right) = x^\mu(\tau) + \lambda(\tau)\partial_\tau x^\mu(\tau)$$
$$\delta x^\mu(\tau) = \lambda(\tau)\partial_\tau x^\mu(\tau)$$

Now, consider the + component of the translation. From above, we have that $\delta x^+(\tau) = 0$, which means that the translation and reparameterisation cancel exactly. In other words

$$\varepsilon^+ + \lambda\partial_\tau x^+(\tau) = \varepsilon^+ + \lambda\frac{p^+}{m^2} = 0 \Rightarrow \lambda = -\frac{m^2}{p^+}\varepsilon^+$$

And this explains the other components of $\delta x^\mu(\tau)$. In fact, this makes sense – if we'd simply change x^+ by a small amount, x^+ would then violate the light-cone Gauge condition.

Errrrr… How kind of the physics!! Isn't it a bit circular!

- One last comment – it's important to note that the p^+ and p^- **Gauge** operators defined above are different to the $p^{\pm} = \left(p^0 \pm p^1\right)/\sqrt{2}$ defined above. It turns out the commutation relations are similar.

4. Light-cone Lorentz Generators

- We saw that Lorentz translations are given by $\delta x^\mu(\tau) = \varepsilon^{\mu\nu} x_\nu(\tau)$, with $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$, and with associated Lorentz charges
$$M^{\mu\nu} = x^\mu(\tau)p^\nu(\tau) - x^\nu(\tau)p^\mu(\tau)$$
These Lorentz charges are Hermitian
- The Lie Algebra of Lorentz generator is defined by
$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\mu\sigma}M^{\rho\nu} - i\eta^{\nu\sigma}M^{\rho\mu}$$
In any coordinate system we choose, the Lorentz generators will have to fulfil these conditions
- We now need to find the generators in light cone *coordinates* (not the light-cone components of the Gauge invariant coordinates)

Chapter 12 – The Relativistic Quantum Open String

1. Light-cone Hamiltonian and Commutators

- We found a class of world-sheet parameterisations for which the equations of motion were wave equations $\ddot{X}^\mu - X^{\mu\prime\prime} = 0$.

  I don't get it – didn't we have those before?

- These come at the expense of constraints $\left(\dot{X} \pm X'\right)^2 = 0$, with which we get

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X^{\mu\prime} \qquad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu$$

- These work in all the Gauges of the class we have considered, but particularly with the light-cone gauge, for which $X^+ = 2\alpha' p^+ \tau$. We then solved for X^- and found that

$$\dot{X}^- = \frac{1}{2\alpha'}\frac{1}{2p^+}\left(\dot{X}^I\dot{X}^I + X^{I\prime}X^{I\prime}\right)$$

  Which gives us, explicitly

$$\mathcal{P}^{\tau -} = \frac{\pi}{2p^+}\left(\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + \frac{X^{I\prime}X^{I\prime}}{\left(2\pi\alpha'\right)^2}\right)$$

- We next choose operators for our theory

$$X^I \qquad x_0^- \qquad \mathcal{P}^{\tau I} \qquad p^+$$

  Sensible commutation relations are

$$\left[X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')\right] = i\eta^{IJ}\delta\left(\sigma - \sigma'\right)$$
$$\left[x_0^-, p^+\right] = -i$$

  With all other commutators vanishing. (Note that x_0 and p^+ do not depend on τ).

- A Hamiltonian that makes sense (since we know p^- generates X^+ translations and that $X^+ = 2\alpha' p^+ \tau$) is

$$\begin{aligned} H(\tau) &= 2\alpha' p^+ p^- \\ &= \pi\alpha' \int_0^\pi \mathrm{d}\sigma \left(\mathcal{P}^{\tau I}(\tau,\sigma)\mathcal{P}^{\tau I}(\tau,\sigma) + \frac{X^{I\prime}(\tau,\sigma)X^{I\prime}(\tau,\sigma)}{\left(2\pi\alpha'\right)^2}\right)\end{aligned}$$

This is sort of equal to $L_0^\perp$, but not quite, for reasons we'll see later.

- The classical boundary conditions become operator equations

$$\partial_\sigma X^I(\tau,\sigma) = 0 \qquad \sigma = 0,\pi$$

Means that the operator $\partial_\sigma X^I$ actually vanishes at the endpoints.

- We can also find the following commutators

$$\left[\left(\dot{X}^I \pm X^{I\prime}\right)(\tau,\sigma),\left(\dot{X}^J \pm X^{J\prime}\right)(\tau,\sigma)\right] = \pm 4\pi\alpha' i\eta^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta\left(\sigma-\sigma'\right)$$
$$\left[\left(\dot{X}^I \pm X^{I\prime}\right)(\tau,\sigma),\left(\dot{X}^J \mp X^{J\prime}\right)(\tau,\sigma)\right] = 0$$

2. Commutation relations for oscillators

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Chapter 13 – Relativistic Quantum Closed Strings

1. Mode Expansions and

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Chapter 14 – Relativistic Superstrings

- **Introduction**
  - Two operators that anticommute satisfy $b_1 b_2 = -b_2 b_1 \Rightarrow \{b_1, b_2\} = 0$.
  - Two *variables* that anticommute satisfy $b_1 b_1 = -b_1 b_1 \Rightarrow b_1 = 0$.
  - To describe the relativistic electron, we use the **Dirac Field** (a classical anticommuting field variable). This leads to **creation operators** $\hat{f}^\dagger_{p,s}$, labelled by momentum and spin. They anticommute, and so $\hat{f}^\dagger_{p,s} \hat{f}^\dagger_{p,s} = 0$, which automatically encodes Pauli's Exclusion principle
- **World-Sheet Fermions**
  - For **Bosonic strings**, we used X^μ – variables that classically commute.
  - For **Fermionic strings**, we'll use new dynamic variables, $\psi^\mu_\alpha(\tau, \sigma)$, where $\alpha = 1,2$.
  - The light-cone gauge now sets $\psi^+_\alpha = 0$ and both the X^- and ψ^-_α receive contributions from the transverse X^I and ψ^I_α.
  - By using the Dirac action and all kinds of weird and wonderful math, we end up with
    $$(\partial_t + \partial_\sigma)\psi^I_1 = 0 \qquad (\partial_t - \partial_\sigma)\psi^I_2 = 0$$
    And boundary conditions
    $$\psi^I_1(\tau,\sigma_*)\left(\delta\psi^I_1(\tau,\sigma_*)\right) - \psi^I_2(\tau,\sigma_*)\left(\delta\psi^I_2(\tau,\sigma_*)\right) = 0$$
    At the endpoints, $\sigma_* = 0, \pi$.
  - From this, we can deduce lots of things
    - The ψ^I_α fields are anticommuting.
    - ψ^I_1 is **right-moving** and ψ^I_2 is **left-moving**.
      $$\psi^I_1(\tau,\sigma) = \Psi^I_1(\tau - \sigma)$$
      $$\psi^I_2(\tau,\sigma) = \Psi^I_2(\tau + \sigma)$$
    - The boundary conditions require that $\psi^I_1(\tau,\sigma_*) = \pm\psi^I_2(\tau,\sigma_*)$. The choice is irrelevant, and so

- We declare that $\boxed{\psi_1^I(\tau,0) = \psi_2^I\left(\tau,0\right)}$
- This makes the sign at the other end relevant $\boxed{\psi_1^I\left(\tau,\pi\right) = \pm\psi_2^I\left(\tau,\pi\right)}$

This divides the string into two **sectors**. The **top sign** is the **Ramond (R) Sector** and the **bottom sign** is the **Neveu-Schwarz (NS) Sector**.

- In fact, we define

$$\Psi^I(\tau,\sigma) = \begin{cases} \psi_1^I(\tau,\sigma) & \sigma \in \left[0,\pi\right] \\ \psi_2^I(\tau,-\sigma) & \sigma \in \left[-\pi,0\right] \end{cases}$$

Notes:

- The boundary condition at $\sigma_* = 0$ ensures that it's continuous.
- The left-moving and right-moving conditions imply that $\boxed{\Psi^I(\tau,\sigma) = \chi^I(\tau-\sigma)}$
- The other boundary condition implies that $\boxed{\Psi^I\left(\tau,\pi\right) = \pm\Psi^I(\tau,-\pi)}$. So **periodic fermions** correspond to **Ramond BCs** and **antiperiodic fermions** corresponds to **Neveu-Schwarz BCs**.

- <u>**Neveu-Schwarz Sector**</u>
  - The Neveu-Schwarz fermion changes sign when $\sigma \to \sigma + 2\pi$, and so it must be expanded with **fractionally moded exponentials**

$$\Psi^I(\tau,\sigma) \sim \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^I e^{-ir\left(\tau-\sigma\right)}$$

Stuff about the coefficients

- They're **anticommuting**

$$\boxed{\left\{b_r^I, b_s^J\right\} = \delta_{r+s,0}\delta^{IJ}}$$

- The…
  - **Negatively moded coefficients** $b_{-1/2}^I, b_{-3/2}^I, \cdots$ are **creation operators**.

- **Positively moded coefficients** $b^I_{1/2}, b^I_{3/2}, \cdots$ are **annihilation operators**.
  - These operators act on the **Neveu-Schwarz vacuum** $\left|\text{NS}\right\rangle$
- Because the X are still quantised as usual, the states are

$$\left|\lambda\right\rangle = \prod_{I=2}^{9}\prod_{n=1}^{\infty}\left(\alpha^I_{-n}\right)^{\lambda_{n,I}} \prod_{J=2}^{9}\prod_{r=\frac{1}{2},\frac{3}{2},\cdots}\left(b^J_{-r}\right)^{\rho_{r,J}}\left|\text{NS}\right\rangle \otimes \left|p^+, \boldsymbol{p}_T\right\rangle$$

Notes:
  - The order of the b matters not, because changing the order will only change overall sign.
  - The ρ must be 0 or 1, because the b anticommute and so $bb = 0$.
- The mass squared operator is (using full ordering)

$$M^2 = \frac{1}{\alpha'}\left(-\frac{1}{2} + N^\perp\right) \qquad N^\perp = \sum_{p=1}^{\infty}\alpha^I_{-p}\alpha^I_p + \sum_{r=\frac{1}{2},\frac{3}{2},\cdots} r b^I_{-r} b^I_r$$

The eigenvalue of $N^\perp$ on $b^I_{-r_1} b^J_{-r_2}\left|\text{NS}\right\rangle$ is $r_1 + r_2$.

- <u>**The *F* number**</u>
  - We define an operator $\left(-1\right)^F$, which is +1 for bosonic states, and –1 for fermionic states. F is the **fermion number**.
  - We first declare that the **vacuum states** are **Fermionic**

$$\left(-1\right)^F\left|\text{NS}\right\rangle \otimes \left|p^+, \boldsymbol{p}_T\right\rangle = -\left|\text{NS}\right\rangle \otimes \left|p^+, \boldsymbol{p}_T\right\rangle$$

Acting on the generic state, we then get

$$\left(-1\right)^F\left|\lambda\right\rangle = -\left(-1\right)^{\sum_{r,J}\rho_{r,J}}\left|\lambda\right\rangle$$

This follows if we take

$$\left\{\left(-1\right)^F, b^I_r\right\} = 0$$

  - From this, we get that states with an **even/odd number** of **fermionic oscillators** are **fermions/bosons**.
- <u>**Ramond sector**</u>
  - With **Remond BCs**, the field is periodic, and so we need **integer moded exponentials**

$$\Psi^I(\tau,\sigma) \sim \sum_{n\in\mathbb{Z}} d_n^I e^{-in(\tau-\sigma)}$$

With, as ever, the **negative/positive** modes being **creation/annihilation** operators . Once again

$$\left\{d_m^I, d_n^J\right\} = \delta_{m+n,0}\delta^{IJ}$$

- The eight d_0 operators are difficult to deal with, and give distinct vacuum. It turns out that they can be organised simply by **linear combination** of **four creation operators** ξ_1,ξ_2,ξ_3,ξ_4 and **four annihilation operators**.
  - The zero modes do not contribute to the mass squared.
  - They construct $2^4 = 16$ degenerate Ramond ground states by acting on the vacuum $\left|0\right\rangle$.
  - **Eight** of these states, denoted $\left|R_a\right\rangle$, have an **even number of creation operators**, and the other eight, denoted $\left|R_{\bar{a}}\right\rangle$, have an **odd number of creation operators**.
  - We denote them $\left|R_A\right\rangle$, with $A = 1, \ldots, 16$
- The states in the Ramond sector are then

$$\left|\lambda\right\rangle = \prod_{I=2}^{9}\prod_{n=1}^{\infty}\left(\alpha_{-n}^I\right)^{\lambda_{n,I}}\prod_{J=2}^{9}\prod_{m=1}^{\infty}\left(d_{-m}^J\right)^{\rho_{m,J}}\left|R_A\right\rangle\otimes\left|p^+,\boldsymbol{p}_T\right\rangle$$

  Once again, the ρ are either 0 or 1.
- Once again, we have a $(-1)^F$ operator, and $\left\{\left(-1\right)^F, d_n^I\right\} = 0$. We also declare $\left(-1\right)^F\left|0\right\rangle = -\left|0\right\rangle$, which implies that
  - $\left|R_a\right\rangle$ are **fermionic**
  - $\left|R_{\bar{a}}\right\rangle$ are **bosonic**
- We have

$$M^2 = \frac{1}{\alpha'}\sum_{n\geq 1}\left(\alpha_{-n}^I\alpha_n^I + nd_{-n}^I d_n^I\right)$$

- We thus have, for each mass level, a Boson and a fermion. This is good – it looks like supersymmetry. But it's only on the worldsheet, not necessarily in spacetime.

- <u>**Generating functions**</u>

- We want to construct generating functions that encode the number of states at any mass levels. We want a function $f(x)$ such that

$$f(x) = \sum_{n=0}^{\infty} a(n) x^n$$

Where $a(n)$ is the number of states with $N^\perp = n$.

- Consider – if we only have one oscillator $a_1^\dagger$, then there is just one state, $\left|0\right\rangle$ with $N^\perp = 0$, and one state $\left(a_1^\dagger\right)^k \left|0\right\rangle$ with $N^\perp = k$. As such, we want

$$f_1(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

If, on the other hand, we have an oscillator with mode 2 (eg: $a_2^\dagger$), we can only get even $N^\perp$, so the function we want is

$$f(x) = 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$$

- It turns out that if we have oscillators $a_1^\dagger, a_2^\dagger, \cdots$, the function is

$$f(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

Similarly, if we have operators of type A that give f_A and operators of type B that give f_B, then the combination will give f_{AB}.

- For example, for our bosonic string theory, we have 24 of each oscillator, and so we have $\prod\left(1-x^n\right)^{-24}$. However, these count the $N^\perp$, and we want the $\alpha' M^2 = N^\perp - 1$ states, so we divide by x and get

$$f_{os}(x) = \frac{1}{x}\prod_{n=1}^{\infty} \frac{1}{\left(1-x^n\right)^{24}} = \frac{1}{x} + 24 + 324x + \cdots$$

Which concurs with our 1 tachionic state, 24 massless Maxwell states, etc…

- What about the fermionic states? If we have a single fermionic operator f_{-r}, we can only get two states: $\left|0\right\rangle$ and $f_{-r}\left|0\right\rangle$, and so $f_r(x) = 1 + x^r$.
  - For the NS sector, each oscillator comes in 8 species, and so

$$\prod_{n=1}^{\infty}\left(1+x^{n-\frac{1}{2}}\right)^8$$

Finally, remembering that $\alpha' M^2 = N^\perp - \frac{1}{2}$, and including the 8 bosonic coordinates that provide $\left(1-x^n\right)^{-8}$, we get

$$f_{\text{NS}}(x) = \frac{1}{\sqrt{x}}\prod_{n=1}^{\infty}\left(\frac{1+x^{n-\frac{1}{2}}}{1-x^n}\right)^8 = \frac{1}{\sqrt{x}} + 8 + 36\sqrt{x} + 128x$$

- For the R sector, we have no offset since $\alpha' M^2 = N^\perp$, and we only have integer oscillators, and so

$$f_{\text{R}}(x) = 16\prod_{n=1}^{\infty}\left(\frac{1+x^n}{1-x^n}\right)^8 + 16 + 256x + 2304x^2 + \cdots$$

We note that the NS functions also include half-integer powers of x, and that the R coefficients are twice the NS coefficients.

- **Open superstrings**
  - The d_0^I have spacetime indices and so transform adequately under Lorentz transformations. The $\left|R_a\right\rangle$, however, do not. In fact, both the $\left|R_a\right\rangle$ and $\left|R_{\bar{a}}\right\rangle$ transform as **spinors**; which is what we need for **spacetime fermions**.
  - However, we do not get two spacetime fermions because (1) the two different states have different values of $(-1)^F$ and so different commuting character (2) we would not get spacetime supersymmetry. Similarly, we cannot identify one as fermions and one as bosons, because bosons cannot carry spinor indices.
  - Thus, we **truncate** the R sector into the R– sector (with $(-1)^F = -1$) and the $R+$ sector. The generating functions for each are

$$f_{R-}(x) = 8\prod_{n=1}^{\infty}\left(\frac{1+x^n}{1-x^n}\right)^8$$

  - Now, for the NS sector.
    - The ground states are tachyonic with $(-1)^F = -1$.

- We define the NS+ sector to only keep states with $(-1)^F = +1$. These have an **odd** number of oscillators and so even mass squared values.
- To find a generating function from this sector, we note that flipping a sign as follows

$$f_{\mathrm{NS}}(x) = \frac{1}{\sqrt{x}}\prod_{n=1}^{\infty}\left(\frac{1+x^{n-\frac{1}{2}}}{1-x^n}\right)^8 \to \frac{1}{\sqrt{x}}\prod_{n=1}^{\infty}\left(\frac{1-x^{n-\frac{1}{2}}}{1-x^n}\right)^8$$

Only flips the sign which have an *odd* number of Fermions. Thus, we need to subtract this to the original expression and divide by two

$$f_{\mathrm{NS}+}(x) = \frac{1}{2\sqrt{x}}\left\{\prod_{n=1}^{\infty}\left(\frac{1+x^{n-\frac{1}{2}}}{1-x^n}\right)^8 - \prod_{n=1}^{\infty}\left(\frac{1-x^{n-\frac{1}{2}}}{1-x^n}\right)^8\right\}$$

For supersymmetry, we require $f_{\mathrm{NS}+}(x) = f_{\mathrm{R}-}(x)$, an identity with was proved by Jacobi.

- **Closed string theories**
  - Closed strings are obtained by combining right-movers and left-movers. We can choose a sector for each copy, and we get four combinations (L, R) = (NS, NS), (NS, R), (R, NS), (R, R).
    - **Bosons** arise from the (NS, NS) and (R, R) [doubly fermionic] sectors.
    - **Fermions** arise from the mixed sectors.
  - To get supersymmetry, we have to truncate each of the sectors. Several options are possible
    - **Type IIA superstrings**: always choose {L} = {NS+, R–} and {R} = {NS+, R+}. This gives

(NS+, NS+) (NS+, R+) (R–, NS+) (R–, R+)

With masses $\frac{1}{2}\alpha' M^2 = \alpha' M_L^2 + \alpha' M_R^2$, where the leval-matching condition ensures that the contribution from both sides match. The massless states are obtained by combining the other various massless states of the different sectors

$$
\begin{array}{llll}
(\text{NS+,NS+}): & \bar{b}^{I}_{-1/2}\left|\text{NS}\right\rangle_L & \otimes b^{J}_{-1/2}\left|\text{NS}\right\rangle_R & \otimes\left|p^+,\boldsymbol{p}_T\right\rangle \\
(\text{NS+,R+}): & \bar{b}^{I}_{-1/2}\left|\text{NS}\right\rangle_L & \otimes\left|R_{\bar{b}}\right\rangle_R & \otimes\left|p^+,\boldsymbol{p}_T\right\rangle \\
(\text{R}-,\text{NS}+): & \left|R_b\right\rangle_L & \otimes b^{I}_{-1/2}\left|\text{NS}\right\rangle_R & \otimes\left|p^+,\boldsymbol{p}_T\right\rangle \\
(\text{R}-,\text{R}+): & \left|R_a\right\rangle_L & \otimes\left|R_{\bar{b}}\right\rangle_R & \otimes\left|p^+,\boldsymbol{p}_T\right\rangle
\end{array}
$$

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<u>Questions to ask tomorrow</u>

- What do the "r" denote in the b operators
- On the top of page 323, why do we sometimes use R_a and sometimes R_b?
- On top of page 314, I'm uncomfortable with the r in the second sum of equation 14.37
- Why does is $\left|\text{NS}'\right\rangle$ bosonic, but the $\left|\text{NS}\right\rangle$ fermionic?
- Why do we keep R'+ and NS'+, but NS+ and R-
- In the heterotic SO(32) sting theory, I don't get why we don't combine *any* of the left ones with *any* of the right ones…
-
- Page 258, at the bottom – how is that implied?
- Why is the state space defined with an a in chapter 12 and with an alpha in chapter 14?
- $\alpha_0^I \propto \boxed{p^I}$ is a momentum operator and annihilates the vacuum states which hav no momentum. But they commute with everything, so what *don't* they annihilate? How can we get a state with momentum?