# 8.01 Review Session

## ***Problem Solutions***

Hi all,

I'm sending round these solutions in case you weren't able to get down some of the stuff I wrote on the boards at last night's review session. Two points:

- The solutions are not nearly as detailed as I usually make solutions. Most importantly, I did not spend a long time explaining tricky leaps of logic, since they were explained in last night's review.
- I avoided using $\ddot{x}$ notation in these solutions. Instead, I wrote $\ddot{x} = a$ and $\dot{x} = v$.

I hope these help, and best of luck for the exam!

Daniel

PS: Feedback on these solutions and/or on the review session are always appreciated.

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**Problem 1**

We define $\boldsymbol{r}$ are the outwards radial direction, and $\boldsymbol{k}$ as the upwards $z$-direction.

This is a system in which there are two contributions to the angular momentum – the wheel is rotating about its *own* axis perpendicular to its centre of mass, and is also precessing about an axis parallel to its diameter, but a distance $R$ away.

Let's work out each of the angular momenta

- **The precession angular momentum**
  - It will be pointing **upwards**, by the right-hand-rule ($+\boldsymbol{k}$ direction).
  - The moment of inertia about that axis, by the parallel axis theorem, will be

$$I_{\text{precess}} = I_d + MR^2$$

  - And therefore

$$\boldsymbol{L}_{\text{precess}} = I_{\text{precess}} \Omega \hat{z}$$

$$\boxed{\boldsymbol{L}_{\text{precess}} = \left(I_d + MR^2\right)\Omega \hat{\boldsymbol{k}}}$$

- **The rotational angular momentum**
  - It will be point **inwards**, by the right hand rule ($-\boldsymbol{r}$ direction).
  - Therefore,

$$\boldsymbol{L}_{\text{rotation}} = -I_0 \omega \hat{\boldsymbol{r}}$$

  - But the question doesn't tell us what $\omega$ is!
  - However, we can deduce that – because the length of the arm, $R$, is the same as the radius of the wheel $R$, we must have $\Omega = \omega$.
  - Therefore

$$\boxed{\boldsymbol{L}_{\text{rotation}} = -I_0 \Omega \hat{\boldsymbol{r}}}$$

The total angular momentum will therefore be

$$\boldsymbol{L}_{\text{total}} = \boldsymbol{L}_{\text{precess}} + \boldsymbol{L}_{\text{rotation}}$$

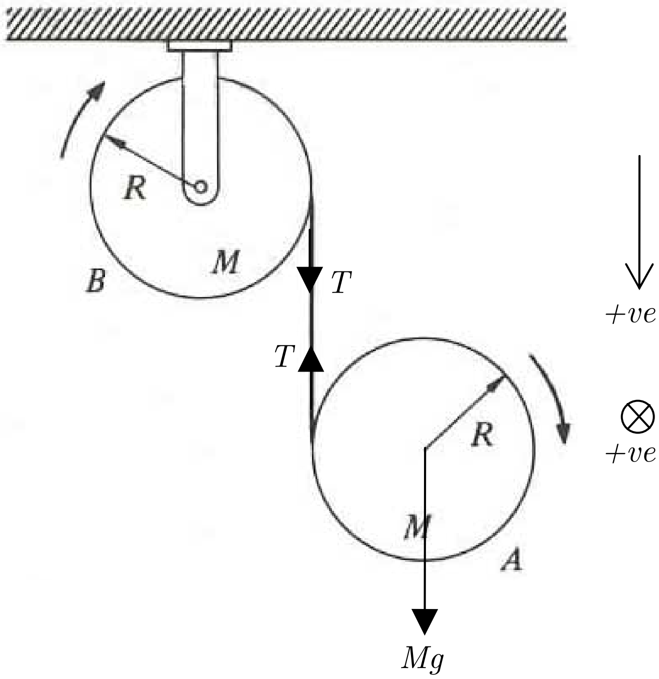
$$\boldsymbol{L}_{\text{total}} = \left(I_d + MR^2\right)\Omega \hat{\boldsymbol{k}} - I_0 \Omega \hat{\boldsymbol{r}}$$

Putting in expressions for $I_d$ and $I_0$, we get

$$\boxed{\boldsymbol{L}_{\text{total}} = MR^2\Omega\left(\frac{3}{2}\hat{\boldsymbol{z}} - \hat{\boldsymbol{r}}\right)}$$

**Problem 2**

- Clearly, this is a dynamics problem.
- Let's look at each bit of our system – where do we have rotational motion and were do we have translational?
    - **Rotational** – both drums
    - **Translational** – the bottom drum
- So we'll have three equations of motion.
- Let's choose a coordinate system and indicate the forces involved:




- So, our three equations of motion are

$$Ma_A = Mg - T \quad (1)$$

$$I\alpha_B = RT \quad (2)$$

$$I\alpha_A = RT \quad (3)$$

- The linking equation is a bit more tricky. If drum $A$ falls, it can be due **either** to the string unwinding from $A$ **or** to the string unwinding from $B$. Therefore

$$y_A = R\theta_A + R\theta_B$$
$$v_A = R\omega_A + R\omega_B$$
$$a_A = R\left(\alpha_A + \alpha_B\right) \quad (4)$$

- And now we just need to solve for $a_A$:

- Start off with (4), and get expressions for $\alpha_A$ and $\alpha_B$ from (2) and (3) and put them in

$$a_A = R\left(\alpha_A + \alpha_B\right)$$
$$a_A = R\left(\frac{RT}{I} + \frac{RT}{I}\right)$$
$$a_A = \frac{2R^2}{I}T$$

- Get an expression for $T$ from equation (1) and re-arrange

$$a_A = \frac{2R^2}{I}\left(Mg - Ma_A\right)$$
$$a_A = \frac{2R^2 Mg}{I} - \frac{2R^2 M}{I}a_A$$
$$\left(1 + \frac{2R^2 M}{I}\right)a_A = \frac{2R^2 Mg}{I}$$
$$\left(\frac{2R^2 M + I}{I}\right)a_A = \frac{2R^2 Mg}{I}$$
$$\boxed{a_A = \frac{2R^2 Mg}{2R^2 M + I}}$$

Which is the answer we want.

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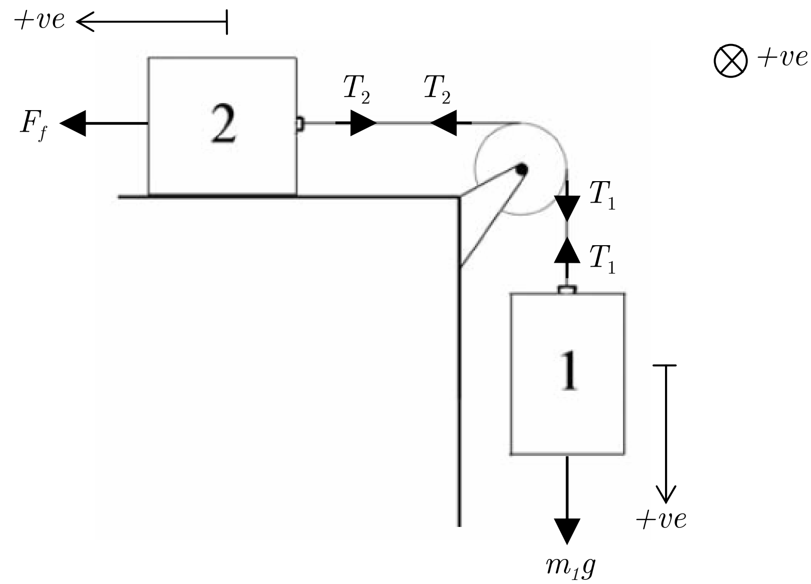
**Problem 3**

- Clearly, we have a dynamics problem again, with two linear motions (each block) and one angular motion (the pulley).
- Forces and coordinate system are as follows: (Note: this is <u>not</u> the best coordinate system to choose – I chose it this way to show you how to cope with choosing a bad coordinate system):




- Let's rock and roll

$$m_2a_2 = F_f - T_2 = \mu m_2 g - T_2 \tag{1}$$

$$m_1a_1 = m_1g - T_1 \tag{2}$$

$$I\alpha = RT_1 - RT_2 \tag{3}$$

- Linking equations [note the negative sign, because of the coordinate system we chose]:

$$a_1 = -a_2 \tag{4}$$

$$a_1 = R\alpha \tag{5}$$

- $a_1$ is what we want – solve to find it
  - Start off with (5), and get $\alpha$ from (3) and substitute it in

$$a_1 = R\frac{\left(RT_1 - RT_2\right)}{I}$$

$$a_1 = \frac{R^2}{I}\left(T_1 - T_2\right)$$

  - Get $T_1$ and $T_2$ from (1) and (2) and re-arrange

$$a_1 = \frac{R^2}{I}\left(m_1 g - m_1 a_1 + m_2 a_2 - \mu m_2 g\right)$$
$$a_1 = \frac{R^2}{I}\left(m_1 g - m_1 a_1 - m_2 a_1 - \mu m_2 g\right)$$
$$a_1 = \frac{R^2}{I} m_1 g - \frac{R^2}{I} m_1 a_1 - \frac{R^2}{I} m_2 a_1 - \frac{R^2}{I} \mu m_2 g$$
$$\left(\frac{R^2(m_1 + m_2) + I}{I}\right) a_1 = \frac{R^2 g(m_1 - \mu m_2)}{I}$$
$$a_1 = \frac{R^2 g(m_1 - \mu m_2)}{R^2(m_1 + m_2) + I}$$

- Substitute $I = MR^2/4$ for a wheel:

$$\boxed{a_1 = \frac{g(m_1 - \mu m_2)}{m_1 + m_2 + \frac{1}{4} m_p}}$$

- This is constant. So to find how far it's travelled after a time $t_1$, use

$$x = v_0 t + \tfrac{1}{2} a t^2$$
$$x = \tfrac{1}{2} a t_1^2$$
$$\boxed{x = \frac{g(m_1 - \mu m_2)}{2m_1 + 2m_2 + \frac{1}{2} m_p} t_1^2}$$

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**Problem 4**

- Clearly, this is a "before an after" type problem.
- There are two choices for the system, though.
- **Option 1** – choose bullet + rod as the system
  - External impulse = 0
  - Initial ang. mom = final ang. mom
  - Take angular momentum about pivot point (remember – to conserve angular momentum, you must take moments about a fixed point).
  - $L_{\text{initial}} = mvL$
  - $L_{\text{final}} = I\omega$
  - So

$$mvL = I\omega$$
$$\omega = \frac{mvL}{I}$$
$$\boxed{\omega = \frac{3mv}{ML}}$$

- **Option 2** – choose rod as system
  - Angular impulse $J$ on rod causes it to move.
  - $L_{\text{initial,rod}} = 0$, $L_{\text{final,rod}} = I\omega$
  - Therefore, $I\omega = J$.
  - To find $J$, note that **linear** impulse on bullet must have been $mv$ (because its angular momentum was changed by $mv$).
  - Therefore, the resulting angular impulse was $mvL$.
  - As such, $mvL = I\omega$, as above.

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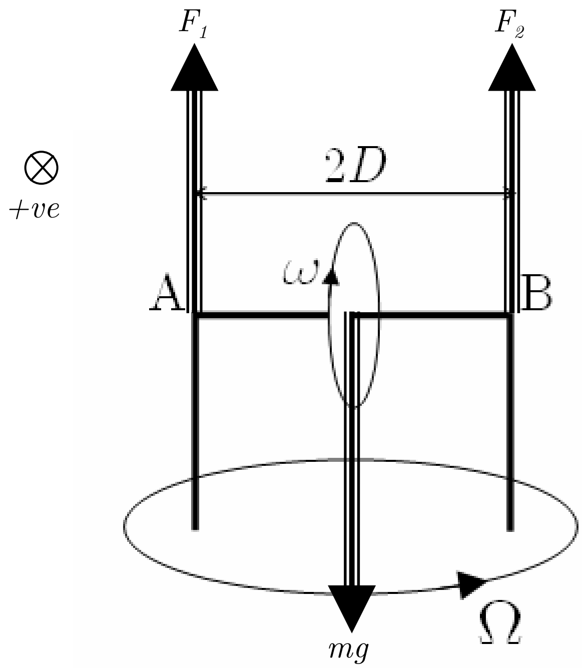
**Problem 5**

- The direction of $L$ is changing, so this is a gyroscopic motion problem.
- Let's draw the forces in for our system:




- First, we note that we have an angular momentum due to the wheel, $L = I_0\omega$ pointing to the **left**.
- This angular momentum is precessing horizontally at a speed $\Omega$. So, about the centre of the wheel,

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \tau = I\omega\Omega \text{ [Out of the page]}$$

[Looking at the precession direction shows that the change is clearly out of the page – the $\boldsymbol{L}$ vector is *to the left* and as the turntable precesses, comes *towards us*].

- And now, let's see what we get from the free-body diagram

$$F_1 + F_2 = mg$$
$$\tau = F_1 D - F_2 D = -I\omega\Omega$$

[Note the negative sign in $\mathrm{d}\boldsymbol{L}/\mathrm{d}t$, because of the sign convention we chose above – into the page we said was positive, so out of the page is negative].

- Solving simultaneously

$$\left(mg - F_2\right)D - F_2 D = -I\omega\Omega$$
$$mgD - 2F_2 D = -I\omega\Omega$$
$$mgD + I\omega\Omega = 2F_2 D$$
$$\boxed{F_2 = \frac{mgD + I\omega\Omega}{2D}}$$

And

$$F_1 = mg - F_2$$
$$F_1 = mg - \frac{mgD + I\omega\Omega}{2D}$$
$$F_1 = \frac{2Dmg - mgD - I\omega\Omega}{2D}$$
$$\boxed{F_1 = \frac{mgD - I\omega\Omega}{2D}}$$

- We now have two expressions. If we plug in numbers and we find that one of them turns out to be negative, it just means that the force was actually pointing downwards instead of upwards.

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**Problem 6**

**Part A**

The spin angular momentum of the wheel points upwards and diagonally right (by the right hand rule) and has magnitude $I_0\omega$. Resolving it into components, we see that it is given by

$$L_{\text{spin}} = I_0\omega\left(\hat{x}\cos 45 + \hat{y}\sin 45\right) = \frac{I_0\omega}{\sqrt{2}}\left(\hat{x}+\hat{y}\right)$$

For the "retracting" part of the angular momentum, we have a disc rotating about the pivot, about an axis parallel to the diameter of the disc. The moment of inertia of the disc about that point is $I = I_{\text{d}} + MD^2$. The angular momentum there points directly along the positive $z$ direction, and so

$$L_{\text{retracting}} = \left(I_{\text{d}} + MD^2\right)\Omega\hat{z}$$

And so the total angular momentum is

$$L_{\text{total}} = \left(I_{\text{d}} + MD^2\right)\Omega\hat{z} + \frac{I_0\omega}{\sqrt{2}}\left(\hat{x}+\hat{y}\right)$$

**Part B**

Only the non-$z$ component is changing, and is rotating at a velocity $\Omega$ and so

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \left|\boldsymbol{L}\right|\left|\Omega\right|\frac{\hat{\boldsymbol{y}} - \hat{\boldsymbol{x}}}{\sqrt{2}}$$

$$\boxed{\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{I_0\omega\Omega}{\sqrt{2}}\left(\hat{\boldsymbol{y}} - \hat{\boldsymbol{x}}\right)}$$

[The $\sqrt{2}$ is there to ensure the vector is normalised].

This is equal to the **total torque** on the system – in other words, from the free-body diagram:

$$\frac{I_0\omega\Omega}{\sqrt{2}}\left(\hat{y}-\hat{x}\right) = \tau_{\text{bearing}} + \tau_{\text{weight}}$$

The torque due to weight is into the page

$$\tau_{\text{weight}} = MgD\sin 45 = -\frac{MgD}{\sqrt{2}}\hat{z}$$

And so

$$\frac{I_0 \omega \Omega}{\sqrt{2}} \left( \hat{y} - \hat{x} \right) = \tau_{\text{bearing}} - \frac{MgD}{\sqrt{2}} \hat{z}$$

$$\boxed{\tau_{\text{bearing}} = \frac{I_0 \omega \Omega}{\sqrt{2}} \left( \hat{y} - \hat{x} \right) + \frac{MgD}{\sqrt{2}} \hat{z}}$$

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**Problem 7**

- This is clearly an energy-angular momentum sort of problem, because we're talking about the two extreme points.
- The orbit is originally circular, with radius $\alpha r$ and eventually elliptical, with maximum radius $\alpha R_{\text{moon}}$ and minimum radius $R_{\text{moon}}$.
- First thing to do, therefore, is find the energy and angular momentum in the **new** orbit. How do we do that? We first need to find the velocity in the old orbit…
- In the old orbit, we had **circular motion** – the velocity of this motion can easily be found

$$F = \frac{mv^2}{r}$$
$$\frac{GM_{\text{moon}}m}{r^2} = \frac{mv^2}{r}$$
$$v_{\text{orig}} = \sqrt{\frac{GM_{\text{moon}}}{r}}$$

And we know that the original radius was $\alpha r$, so

$$v_{\text{orig}} = \sqrt{\frac{GM_{\text{moon}}}{\alpha R_{\text{moon}}}}$$

- After the rocket has fired and the velocity has been halved, the velocity is

$$v_{\text{far}} = \frac{1}{2}\sqrt{\frac{GM_{\text{moon}}}{\alpha R_{\text{moon}}}}$$

- So now, at this point, energy and angular momentum are

$$E = \frac{1}{2}mv^2 - \frac{GM_{\text{moon}}m}{\alpha R_{\text{moon}}} = \frac{GM_{\text{moon}}m}{8\alpha R_{\text{moon}}} - \frac{GM_{\text{moon}}m}{\alpha R_{\text{moon}}}$$
$$L = m\alpha R_{\text{moon}}v = \frac{m}{2}\alpha R_{\text{moon}}\sqrt{\frac{GM_{\text{moon}}}{\alpha R_{\text{moon}}}} = \frac{m}{2}\sqrt{GM_{\text{moon}}\alpha R_{\text{moon}}}$$

- Now, we switch to the other point – we know the radius there (it's $R_{\text{moon}}$) and if its velocity there is $v$, we can say that the energy and angular momentum there are

$$E = \frac{1}{2}mv^2 - \frac{GM_{\text{moon}}m}{R_{\text{moon}}}$$
$$L = mvR_{\text{moon}}$$

- Set them equal to each other

$$\frac{GM_{\text{moon}}m}{8\alpha R_{\text{moon}}} - \frac{GM_{\text{moon}}m}{\alpha R_{\text{moon}}} = \frac{1}{2}mv^2 - \frac{GM_{\text{moon}}m}{R_{\text{moon}}}$$

$$\frac{m}{2}\sqrt{GM_{\text{moon}}\alpha R_{\text{moon}}} = mvR_{\text{moon}}$$

- Clearly, $v$ is something we don't know, so we want to eliminate it. Rearranging, the second equation becomes

$$v^2 = \frac{GM_{\text{moon}}\alpha}{4R_{moon}}$$

Eliminating $v$

$$\frac{GM_{\text{moon}}}{8\alpha R_{\text{moon}}} - \frac{GM_{\text{moon}}}{\alpha R_{\text{moon}}} = \frac{1}{2}\frac{GM_{\text{moon}}\alpha}{4R_{moon}} - \frac{GM_{\text{moon}}}{R_{\text{moon}}}$$

$$\frac{1}{8\alpha} - \frac{1}{\alpha} = \frac{\alpha}{8} - 1$$

$$1 - 8 = \alpha^2 - 8\alpha$$

$$\alpha^2 - 8\alpha + 7 = 0$$

$$\alpha = \frac{8 \pm \sqrt{64 - 28}}{2}$$

$$\alpha = 7 \qquad \text{or} \qquad \alpha = 1$$

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**Problem 8**

**Part A**

Potential is given by

$$U(r) = -\int_0^r -br^3 \,\mathrm{d}r = \frac{1}{4}br^4$$

And so the effective potential is given by

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} + \frac{1}{4}br^4$$

The energy corresponding to circular motion is one that grazes the lowest point.

**Part C**

The effective potentials at the maximum and minimum radius points are equal, and so

$$U_{\text{eff}}(2r_0) = U_{\text{eff}}(r_0)$$
$$\frac{L^2}{2m\left(2r_0\right)^2} + \frac{1}{4}b\left(2r_0\right)^4 = \frac{L^2}{2mr_0^2} + \frac{1}{4}br_0^4$$
$$\frac{L^2}{2mr_0^2} + 16br_0^4 = \frac{2L^2}{mr_0^2} + br_0^4$$
$$15br_0^4 = \frac{3L^2}{2mr_0^2}$$
$$r_0^6 = \frac{1}{10}\frac{L^2}{mb}$$