The Path Integral Formulation of Quantum Mechanics

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8.06 Term Paper
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We discuss the Lagrangian formalism in classical mechanics and the path integral formalism in quantum mechanics, thus providing a unified framework through which both quantum and classical phenomena can be understood. We then explore the path integral formulation in more detail, and show how it can be used to deal with a selection of problems: the Aharonov-Bohm effect, the free particle, the simple harmonic oscillator and perturbation theory.

I. INTRODUCTION

In today's undergraduate courses, mechanics and quantum mechanics are taught very differently. Classical mechanics is usually approached from the perspective of Newton's Laws [1]. Quantum Mechanics is approached using the Schrodinger Equation [2]. These two approaches are radically different, both in underlying approach and methodology, and the kinds of problems studied in each case are wildly different. Furthermore, though the quantum and classical theories can be shown to be consistent [2], there is no obvious way to recover one from the other.

The aim of the first part of this paper is to explore formulations of mechanics and quantum mechanics which make the similarities between the theories much more obvious. In so doing, we will introduce Lagrangians in classical mechanics and path integrals in quantum mechanics. We will also show how Newton's Laws and Schrodinger's Equation arise naturally out of this formalism. In the second part of this paper, we will explore the path integral approach in more detail and show how it can be used to deal with a wide range of problems.

II. THE PROPAGATOR

We begin by introducing the *propagator* – a function of four variables

$$K(x, t \leftarrow x', t')$$

The propagator is such that KK^* gives the *probability* that a system at position x' at time t' will end up at position x at time t (the use of the arrow in denoting the propagator is non-standard, but we find it useful to make it clear the propagator denotes the movement *from* the primed state *to* the unprimed state).

The 'aim' of mechanics – classical and quantum – can, effectively, be reduced to finding the propagator for a given system. Indeed, given initial conditions, the propagator tells us everything we need to know about the future state of the system. This approach is attractive for our purposes because it states both the classical and quantum problem in the same way[14].

Clearly, classical propagators will be completely deterministic, and therefore much simpler, than their quantum counterparts. We will show, however, how classical and quantum propagators can be computed using surprisingly similar principles.

III. THE LAGRANGIAN APPROACH TO CLASSICAL MECHANICS

A. The Lagrangian Method

The Lagrangian approach to classical mechanics prescribes the following method for finding the path of a system from a point A to a point B [3][15]

- Parameterize the system using a set of coordinates. For example, to parameterize a particle moving in one dimension, a single coordinate is needed – the displacement of the particle from an origin (denoted x).

  In fact, a full parameterization of such a system also requires the *momentum* of the particle (denoted p), but we shall not need it in our discussion.

- Construct a *Lagrangian* for the mechanical system, denoted $\mathcal{L}$. This function generally depends on time and on our coordinates and their derivatives. We will see later how to construct this function. In our simple case, we write $\mathcal{L}(x, \dot{x}; t)$.

- Construct the *action* for the mechanical system, denoted S, as follows

  $$S[x] = \int_A^B \mathcal{L}\, \mathrm{d}t$$

  The action is a function of the *path* of a system; given any path $x(t)$ that the particle follows, the action outputs a single number characterising that path.

The Lagrangian approach then states that the path the system *will* follow is the one that *extremises* the action S. In other words – if the path the particle eventually follows is $X(t)$ with an associated action $S[X]$, then every *small* deviation from this path $x(t)$ will result in a larger (or smaller) action $S[x]$.

An important theorem states that this extremal path $x(t)$ satisfies the Euler-Lagrange equation[4]

$$\frac{\partial \mathcal{L}}{\partial x(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}(t)}\right) \tag{1}$$

B. Finding the Lagrangian

All that remains to explain is how to choose the Lagrangian $\mathcal{L}$. The simple answer is – there's no sure way. In the words of Feynman [5]

> The question of what the action should be for a particular case must be determined by some kind of trial and error. It is just the same problem as determining the laws of motion in the first place. You just have to fiddle around with the equations that you know and see if you can get them into the form of a principle of least action.

There are often guiding principles we can use to determine what the Lagrangian *should* look like (Galilean or Lorentz invariance, for example [6]), but it is essentially experimentally determined.

For a mechanical system only involving conservative forces, the non-relativistic classical Lagrangian is given by

$$\mathcal{L} = T(t) - V(x,t) \tag{2}$$

Where T is the kinetic energy of the system and V is the potential energy of the system.

In retrospect, this is sensible, at least for a one-particle system. The total energy of the particle must stay constant, and so the only 'freedom' the particle has is how to distribute this energy amongst its potential and kinetic degrees of freedom. The difference between the kinetic and potential energy is a measure of how the energy is distributed.

C. The Classical Propagator

All that remains is to explain how to find the propagator K. In fact, it seems almost trivial – K is simply equal to 1 if (x,t) and (x',t') lie on a path $x(t)$ that satisfies the Euler-Lagrange equation 1, and 0 otherwise.

IV. THE PATH INTEGRAL FORMULATION OF QUANTUM MECHANICS

We now consider the problem of adapting this approach to quantum problems. Most accounts start from the operator formulation of quantum mechanics and use it to construct the path integral formalism (see section IV D for an outline of this method). We take the slightly lengthier but more insightful approach of starting from the Lagrangian approach of classical mechanics and building in quantum phenomena.

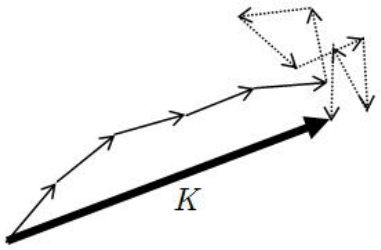

FIG. 1: Each of the small arrows indicate a contribution to the propagator. The contributions have equal magnitude (arrow length) but different phase (arrow direction). The propagator, indicated by the thick arrow is given by the sum of these contributions. The first few contributions (solid arrows) have very similar phases. They therefore contribute a large amount of the propagator. The later contributions (dotted arrows) have very different phases, and tend to cancel each other out; the propagator would not have been very different without those last contributions.

The main issue that arises in doing this is that quantum particles do *not* follow one definite path. We know, in fact, that quantum particles 'spread out'. It therefore makes little sense to ask 'what path does the particle take from point A to point B', because it actually takes *every* path between these two points. A natural corollary of this observation is that a particle starting at point A could end up *anywhere* else.

We must, therefore, talk of the *probability* that a particle will end up in a given place. The problem is therefore naturally stated in terms of the propagator.

It stands to reason that every path to that given place will contribute a certain amount to the probability of ending up there. Feynman postulated that each path contributes an *equal* amount to the propagator, but with a different *phase*.

In fact, he postulated that the full expression for the propagator is [8]

$$K(x,t \leftarrow x',t') \propto \int e^{iS[x(t)]/\hbar}\, \mathscr{D}x \tag{3}$$

Where we have introduced new notation – $\mathscr{D}x$ means 'perform this integral over every possible path from x to x''.

A. Rationalising the Path Integral Formulation

In our discussion of classical mechanics, we postulated that a system will take the path of extremal action because it has no reason to stray from it. In this quantum theory, we now postulate that the system *might* stray

into this region, but only a very small amount. The path integral formulation can be rationalised as an ingenious implementation of this condition.

Consider representing each contribution to the propagator as a vector in phase space. Each vector has identical length, but a different angle (phase), given by $\phi = e^{iS[x(t)]/\hbar}$. The propagator is proportional to the (normalised) vector sum of these contributions.

Now, consider paths (call them x_1 and x_2) close to the classical path (which we will call $\bar{x}$). Because $S[\bar{x}]$ is an extremum of S, $S[x_1]/\hbar$ and $S[x_2]/\hbar$ will not differ appreciably from each other or from $S[\bar{x}]/\hbar$. As a result, the phase ϕ of each of these will be similar, the vectors will be parallel and the vectors will therefore add *in phase*, contributing significantly to the propagator.

Now, consider new paths (call them x_3 and x_4) which are much further away from the classical path. We are now far from an extremum in S, and so $S[x_3]/\hbar$ is likely to be significantly different from $S[x_4]/\hbar$. The phase of these contributions will therefore be very different, and the vectors are therefore likely to 'cancel,' contributing almost nothing to the propagator.

These effects are illustrated in figure 1.

We have therefore seen that the path integral formulation effectively allows us to take into account *every* path the particle might take while heavily biasing the classical paths.

B. The Classical Limit

In fact, a careful scrutiny of the discussion above shows that the 'heavily biased paths' are the ones that are within $\hbar$ of the classical path, because the factor in the exponent is divided by $\hbar$.

This observation makes it clear that the path integral formulation naturally reduces to the classical formulation when the dimensions of the problem become much larger than $\hbar$. As $\hbar$ gets very small, the 'heavily biased paths' become closer and closer to the classical path. Eventually, as $\hbar \to 0$, only the classical path contributes to the propagator, precisely as expected from our discussion in the previous section.

C. Performing the Path Integral

Our expression for the propagator in expression 3 is conceptually very simple. The difficulty arises in evaluating the integral $\mathscr{D}x$ in practice. In this section, we cast the integral in a more familiar form, which will make it easier to evaluate later on in this paper.

Our approach will be to consider two points, x and x', and to split the paths between these two points into N small segments, each spanning a time interval $\epsilon \propto 1/N$. At the start of each segment i, the particle is at position x_i. integrating over *all* possible x_i over *all* space is then equivalent to integrating over every possible path. As $N \to \infty$, the paths become completely smooth. Thus, we can write

$$\int \mathscr{D}x = \lim_{N\to\infty} [C(\epsilon)]^N \int \prod_{i=1}^{N-1} \mathrm{d}x_{i\epsilon} \tag{4}$$

Where C is the constant amount each path contributes to the sum, to be determined (without C, the sum over the infinite number of paths would diverge).

D. Path Integrals from the Operator Picture

As we mentioned, most texts begin the development of the path integral formalism from the operator description of quantum mechanics. We reproduce the first few steps of the method here, for completeness.

Consider the Schrodinger time evolution equation for a wavefunction

$$|\psi, t\rangle = U(t, t') \, |\psi, t'\rangle$$

Now cast this equation in coordinate space, and insert a complete set of position states

$$\begin{aligned} \langle x|\psi, t\rangle &= \int \langle x| \, U(t, t') \, |x'\rangle \, \langle x'|\psi, t'\rangle \, \mathrm{d}x' \\ \psi(x, t) &= \int \langle x| \, U(t, t') \, |x'\rangle \, \psi(x', t') \, \mathrm{d}x' \end{aligned} \tag{5}$$

This integral is over all possible *original states* x' that could lead to the final state x. It seems reasonable, therefore, to say that the first quantity in the integral is equal to the *probability* of going from state x' to state x. A rigorous analysis using Green's Functions [4] confirms our intuition, and in fact

$$\psi(x, t) = \int K(x, t \leftarrow x', t')\psi(x', t') \, \mathrm{d}x' \tag{6}$$

It follows from equations 5 and 6 that

$$K(x, t \leftarrow x', t') = \langle x| \, U(t, t') \, |x'\rangle \tag{7}$$

For a system in which the Hamiltonian is independent of time, this becomes

$$K(x, t \leftarrow x', t') = \langle x| \, e^{-(i/\hbar)H(t-t')} \, |x'\rangle$$

Alternatively

$$K(x, t \leftarrow x', t') = \sum_{n=0}^{\infty} \langle x|E_n\rangle \, e^{-(i/\hbar)(t-t')E_n} \, \langle E_n|x'\rangle \tag{8}$$

This expression for the propagator can then be used, instead of equation 3, to carry out the analyses below. This often involves inserting an infinite set of momentum eigenstates into expression 8. We will leave the interested reader to consult [8] and [9] for more details.

V. NEWTON'S LAWS AND SCHRODINGER'S EQUATION

Before we plunge into application of the path integral formalism, it is worthwhile to spend a second looking back at the formalism we have developed to understand how it relates to Newton's Second Law and Schrodinger's equation.

A. Newton's Second Law

Using the classical Lagrangian for a free particle in equation 2 and feeding it into the Euler-Lagrange Equation 1, we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}\right)$$

We note, however, that $m\dot{x}$ is the expression for Newtonian momentum, and that $\frac{\mathrm{d}V}{\mathrm{d}x} = -F(x)$, where F is the force acting on the particle. This therefore reduces to

$$\frac{\mathrm{d}p}{\mathrm{d}t} = F$$

This is Newton's Second Law.

B. Schrodinger's Equation

Consider equation 6 for the wavefunction, using a very small time increment ϵ

$$\psi(x, t+\epsilon) = \int_{-\infty}^{\infty} K(x, t+\epsilon \leftarrow x', t)\psi(x', t)\ \mathrm{d}x' \quad (9)$$

The propagator in this case is given by equation 3

$$K(x, t+\epsilon \leftarrow x', t) = \int e^{\frac{i}{\hbar}\int \mathcal{L}(x,\dot{x},t)\ \mathrm{d}t}\ \mathscr{D}x$$

We can evaluate this path integral using equation 4. However, the method we used there involves breaking the path into small time slices. In this case, we only have one small time slice ϵ which can be made arbitrarily small. There is therefore no need to split the path any further; only the straight-line path between x and x' matters, and the integrals over intermediate x-values are redundant. The propagator therefore becomes

$$K(x, t+\epsilon \leftarrow x', t) \approx C(\epsilon)\exp\left(\frac{i}{\hbar}\int \mathcal{L}(x, \dot{x}, t)\ \mathrm{d}t\right)$$

Now, consider the straight-line path between x and x'. To first order in ϵ:

- $\int \mathcal{L}\ \mathrm{d}t = \epsilon\mathcal{L}$.
- x takes the average of the values at the endpoints of the interval (ie: $(x+x')/2$)
- $\dot{x}$ takes the value of the gradient of the straight line between the two endpoints of the interval (ie: $(x-x')/\epsilon$)

We therefore get

$$K(x, t+\epsilon \leftarrow x', t) \approx C(\epsilon)\exp\left(\frac{i}{\hbar}\epsilon\mathcal{L}\left(\frac{x+x'}{2}, \frac{x-x'}{\epsilon}\right)\right) \quad (10)$$

Now, consider the example of a single particle moving under the action of a conservative force. From 2, the Lagrangian in this case is

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x,t)$$

We feed this expression into 9, using the simplified propagator in equation 10, and we denote $\Delta = x - x'$

$$\psi(x, t+\epsilon) = \int_{-\infty}^{\infty} C(\epsilon)\exp\left[\frac{i}{\hbar}\frac{m\Delta^2}{\epsilon}\right] \exp\left[-\frac{i}{\hbar}\epsilon V\left(x\frac{\Delta}{2}, t\right)\right]\psi(x+\Delta, t)\ \mathrm{d}\Delta$$

We now need to expand this equation to first order in ϵ. First, however, we note that it is clear that if Δ gets larger than $\sqrt{\epsilon\hbar/m}$, the phase of the first exponent in this expression varies rapidly. Therefore, most of the integral is contributed by values of $\Delta^2 \leq \epsilon\hbar/m$ – outside this range, the rapidly changing phases tend to cancel each other out. As such, when we expand the equation above to first order in ϵ, we need only expand to *second* order in Δ:

$$\psi(x,t) + \epsilon\frac{\partial\psi}{\partial t} = \int_{-\infty}^{\infty} C(\epsilon)\exp\left[\frac{im\Delta^2}{\hbar\epsilon}\right]\left[1 - \frac{i\epsilon}{\hbar}V(x,t)\right] \left[\psi(x,t) + \Delta\frac{\partial\psi}{\partial x} + \frac{1}{2}\Delta^2\frac{\partial^2\psi}{\partial x^2}\right]\ \mathrm{d}\Delta$$

Taking *leading order terms* on both sides of the equation yields

$$C(\epsilon) = \sqrt{\frac{m}{2\pi i\hbar\epsilon}} \quad (11)$$

This is an important result, which we will use later.

Taking *next to leading order* terms on both sides yields

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x,t)\psi$$

Which is precisely Schrodinger's Equation.

VI. APPLICATIONS

We will end by exploring a number of applications of the path integral formulation.

A. The Free Particle

For a free particle, the Lagrangian is $\mathcal{L} = m\dot{x}^2/2$ (see equation 2). For a small time step ϵ starting at x_i and ending at x_j, $\dot{x}$ can be approximated as the gradient of the straight line between these points, and the action for this small step from x_i to x_j is given by

$$S = \int \mathcal{L}\,\mathrm{d}t = \epsilon\frac{1}{2}m\left(\frac{x_j - x_i}{\epsilon}\right)^2$$

The propagator (using equations 3 and 4) is then

$$K = \lim_{\epsilon\to 0} C^{-N/2}\int\prod_{i=1}^{N}\exp\left[\frac{im}{2\hbar\epsilon}(x_i - x_{i-1})^2\right]\mathrm{d}x_N$$

Every integral in this expression is a Gaussian integral, which is easily performed. An expression for the factor C was derived in equation 11. The result of these calculations, after taking the limit, is [8]

$$K(x, t \leftarrow x', t') = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}\exp\left[\frac{im(x-x')^2}{2\hbar(t-t')}\right] \tag{12}$$

This result can be shown to be consistent with the plane-wave energy eigenstates of the free particle

$$\psi(x,t) = Ae^{ipx/\hbar} \tag{13}$$

Using equation 8, and remembering that position eigenstates are Delta-functions ($|x'\rangle = \delta(x - x')$), we obtain

$$K = \int A^2 e^{ipx/\hbar}e^{-(i/\hbar)(t-t')E_p}\left(e^{ipx'/\hbar}\right)^*\,\mathrm{d}E$$

For a free particle, we have $E_p = p^2/2m \Rightarrow \mathrm{d}E_p = \mathrm{d}p/m$

$$K = \frac{A^2}{m}\int\exp\left[\frac{i}{\hbar}(x-x')p - \frac{i}{2m\hbar}(t-t')p^2\right]\,\mathrm{d}p$$

Completing the square in the exponent turns this into a Gaussian integral, which is easily carried out

$$K = \frac{A^2}{m}\sqrt{\frac{2\pi m\hbar}{i(t-t')}}\exp\left[\frac{mi(x-x')^2}{2\hbar(t-t')}\right]$$

This result is consistent with that in equation 12, up to a normalisation constant.

B. The Simple Harmonic Oscillator

A particle undergoing simple harmonic motion moves under the influence of a potential $V(x) = -\frac{1}{2}kx^2$, where k is a constant characterising the harmonic motion. Using equations 2 and 3, we obtain the following expression for the propagator

$$K(x, t \leftarrow x', t') \propto \int\exp\left(\frac{i}{\hbar}\int\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\,\mathrm{d}t\right)\,\mathscr{D}x \tag{14}$$

It can be shown [8][10] that computing this integral gives the following expression for the propagator $K(x, t \leftarrow x', 0)$

$$\sqrt{\frac{m\omega}{ih\sin\omega T}}\exp\left[\frac{im\omega}{2\hbar\sin\omega T}\left[(x')^2 + x^2\right]\cos\omega T - 2x'x\right)\right] \tag{15}$$

The standard way to carry out this computation is somewhat cumbersome. In a paper of 2004 Moriconi [13] proposed a simpler approach, which involves completing the square in equation 14 and using the substitution $T \to iT$.

By expressing the sinusoidal functions in the propagator in terms of exponentials, we can re-write it as

$$\sqrt{\frac{m\omega}{\pi\hbar}}e^{-(i\omega T)/2}\left(1 - e^{-2i\omega T}\right)^{-1/2}\times\cdots$$

Where $\cdots$ is a long and complex expression – the key point, however, is that this expression can be expanded in powers of $e^{-i\omega T}$. Because of the pre-factor of $e^{-i\omega T/2}$, the result will be a power series where each term is of the form

$$e^{-i\omega T/2}e^{-in\omega T} \qquad n = 0, 1, 2, \cdots$$

However, equation 8 states that the propagator has the form

$$K(x, t \leftarrow x', t') = \sum_{n=0}^{\infty}\langle x|E_n\rangle\, e^{-(i/\hbar)(t-t')E_n}\,\langle E_n|x'\rangle$$

Comparing these two equations immediately leads to

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

As previously derived.

A more careful expansion of the $\cdots$ term also leads to an expression for the harmonic oscillator wavefunctions consistent with those previously derived [8].

The path integral formulation can also be used to treat the *forced* quantum harmonic oscillator [8]. The algebraic details are somewhat messy, but the results obtained are of crucial importance in the study of quantum electrodynamics.

C. The Aharonov-Bohm effect

The Aharonov-Bohm effect takes place when a charged particle travels *past* a closed impenetrable region in which a magnetic flux, Φ_B, exists (see figure 2).

Outside this impenetrable region, no magnetic flux exist, but the magnetic vector potential, $\boldsymbol{A}$, is nonzero. This can be shown using the definition of the magnetic vector potential and Stoke's Theorem [9]

$$\begin{aligned}\boldsymbol{B} &= \nabla\times\boldsymbol{A}\\ \int_S\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{A} &= \int_S(\nabla\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{A}\\ \int_S\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{A} &= \oint_C\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{\ell}\end{aligned}$$

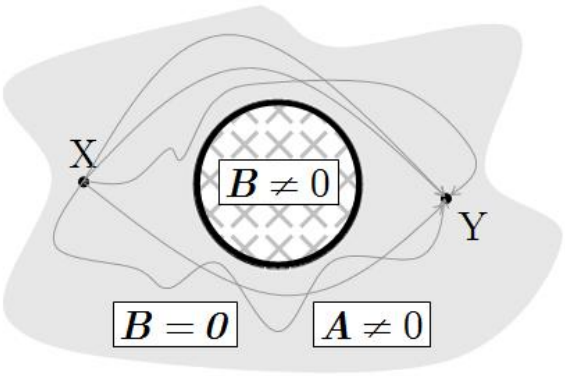

FIG. 2: **The Aharonov-Bohm effect**: A charged particle moving from point X to point Y can take *any* path that does not pass through the central impenetrable region (a few examples are illustrated on the diagram).

Where S is any surface bounded by the curve C. Choosing C to be a circle of radius r concentric with the impenetrable region makes it clear that for every r, $\boldsymbol{A}(r) \neq 0$ at least *somewhere* outside the impenetrable region. Symmetry considerations imply that $\boldsymbol{A}$ can only depend on r, and this therefore implies that $\boldsymbol{A} \neq 0$ *everywhere* outside the impenetrable region.

1. The Relevant Lagrangian

Let $\mathcal{L}_0$ and S_0 denote the Lagrangian and action for a free particle. The classical Lagrangian for a particle in a vector potential $\boldsymbol{A}$ is then given by

$$\mathcal{L} = \mathcal{L}_0 + \frac{e}{c}\dot{\boldsymbol{x}} \cdot \boldsymbol{A} \tag{16}$$

And so the action is given by

$$S = S_0 + \frac{e}{c}\int \boldsymbol{A} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\,\mathrm{d}t$$

We can re-write the second integral as a line integral along the path taken

$$S = S_0 + \frac{e}{c}\int \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{s} \tag{17}$$

2. Addressing the Problem

We are now ready to examine the Aharonov-Bohm effect. Consider a point X to the left of the cylinder, and a point Y to the right of the cylinder. From equation 3, the propagator is given by

$$K = \int \exp\left(\frac{i}{\hbar}S_0 + \frac{ie}{c}\int \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}\right)\ \mathscr{D}x$$

We have defined the problem so that the region of flux is impenetrable. As such, $\mathscr{D}x$ need only be evaluated for paths that pass *above* and *below* the region

$$\begin{aligned} K = &\int_{\text{below}} \exp\left(\frac{i}{\hbar}S_0 + \frac{ie}{c}\int_{\text{below}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}\right)\ \mathscr{D}x \\ &+ \int_{\text{above}} \exp\left(\frac{i}{\hbar}S_0 + \frac{ie}{c}\int_{\text{above}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}\right)\ \mathscr{D}x \end{aligned}$$

Now, the probability of finding the particle at B depends on $|K|^2$ – in other words, it depends on the *amplitude* of K, which depends on the *difference* in phase between the two terms above

$$\begin{aligned} |K|^2 \ &\propto\ \frac{e}{\hbar c}\left[\int_{\text{ab}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} - \int_{\text{bel}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}\right] \\ &=\ \frac{e}{\hbar c}\oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} = \frac{e}{\hbar c}\Phi_B \end{aligned} \tag{18}$$

This result is consistent with results obtained by solving Schrodinger's Equation directly [11], and was experimentally verified by Osakabe et.al. in 1986 using a toroidal magnetic field confined by a superconductor [12].

VII. PERTURBATION THEORY

The path integral formulation also provides a new and insightful way of thinking of perturbation theory.

Consider a particle under the effect of a potential $V(x,t)$. The propagator is given by

$$K_V = \int \exp\left[\frac{i}{\hbar}\int \frac{m}{2}\dot{x}^2\,\mathrm{d}t'\right]\exp\left[-\frac{i}{\hbar}\int V(x,t)\,\mathrm{d}t\right]\ \mathscr{D}x$$

If the time integral of the potential is small with respect to $\hbar$, we can expand the second exponential in this expression as follows

$$1 - \frac{i}{\hbar}\int V(x,t)\,\mathrm{d}t + \frac{1}{2!}\left[\frac{i}{\hbar}\int V(x,t)\,\mathrm{d}t\right]^2$$

Feeding this back into our expression for K_V and expanding, we can write

$$K_V = K_0 + K^{(1)} + K^{(2)} + \cdots$$

Where K_0 is the potential in the absence of a propagator. This result should be familiar from perturbation theory.

We can gain a unique insight, however, into the meaning of the perturbing terms. Consider $K^{(1)}$

$$K^{(1)} = -\frac{i}{\hbar}\int \exp\left[\frac{i}{\hbar}\int \frac{m}{2}\dot{x}^2\,\mathrm{d}t'\right]\left[\int V\left(x(t),t\right)\,\mathrm{d}t\right]\ \mathscr{D}x$$

We can reverse the order of the second time integral and the path integral, to obtain

$$K^{(1)} = -\frac{i}{\hbar}\int F(t)\,\mathrm{d}t \tag{19}$$

$$F(t) = \int \exp\left[\frac{i}{\hbar}\int \frac{m}{2}\dot{x}^2 \, \mathrm{d}t'\right] V\left(x(t),t\right) \mathscr{D}x$$

The expression for $F(t)$ is simply a free particle path integral, with one small modification – it is *weighed* by whatever value the potential has at a the time t. $F(t)$, therefore, represents the particle interacting (scattering) with the potential *a single time*, at a time t, but otherwise travelling freely.

When we find $K^{(1)}$ using equation 19, we integrate over *all possible times* at which this interaction could have happened. As such, $K^{(1)}$ is effectively a term that represents every single way the particle could have interacted *once* with the potential.

A similar analysis of $K^{(2)}$ reveals that it has the same meaning, except that this time, we allow the particle to interact *twice* with the potential.

The perturbation expansion, therefore, can be understood as a sum over all the number of times interactions with the potential can occur. We first find the propagator assuming $V = 0$ (ie: no interaction). We then add on the term corresponding to the particle scattering *once*. Then twice, etc. . . For a weak potential, higher numbers of interactions are more unlikely, and so the first few terms in the expansion are usually sufficient.

This is the mathematical foundation of *Feynman diagrams*, often used to represent interactions of fields and particles in quantum field theory. Diagrams can be drawn either to 'first order', in which a simple, single interaction occurs between fields, or to higher order. Summing all such diagrams is equivalent to summing the terms in a perturbation expansion.

VIII. CONCLUSION

We have explored two new frameworks for quantum and classical mechanics; the Lagrangian formalism and the path integral formalism. In so doing, we were able to base classical and quantum mechanics on a similar theoretical basis. We further showed that both theories reduce to theories we have already encountered, namely those of Newton and Schrodinger.

We briefly looked at some applications of the path integral technique in dealing with the free particle, the harmonic oscillator, the Aharonov-Bohm effect and perturbation theory.

The path integral technique has also been crucial in the development of quantum electrodynamics, which was the first theory to successfully rationalise the divergent integrals that arose in explaining the Lamb shift in hydrogen. It is also the cornerstone of quantum field theory, by virtue of the fact it naturally involves the Lorentz invariant Lagrangian.

[1] See, for example, H.D. Young and R.A. Freedman, *University Physics, 12th Ed.* (Addison Wesley, 2007)

[2] See, for example, A.P. French and E.F. Taylor, *An Introduction to Quantum Physics, 1st Ed.* (CRC, 1979)

[3] See, for example, M. Longair, *Theoretical Concepts in Physics, 3rd Ed.* (Cambridge University Press, 2003)

[4] See, for example, K.F. Riley, M.P. Hobson and S.J. Bence, *Mathematical Methods for Physics and Engineering, 3rd Ed.* (Cambridge University Press, 2006)

[5] R.P. Feynman, *Lectures in Physics*, Vol.2, Chapter 19, (Addison-Wesley)

[6] L.D. Landau and E.M. Lifshitz, *Course of Theoretical Physics: Mechanics, 3rd Ed.* (Butterworth-Heinemann, 1982)

[7] See, for example, M.E. Peskin and D.V. Schroeder *An Introduction to Quantum Field Theory* (Westview Press, 1995).

[8] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, 1965)

[9] See, for example, J.J. Sakurai *Modern Quantum Mechanics* (Benjamin Cummings, 1985)

[10] M.C. Payne and W.J. Stirling *Theoretical Physics 2 Lecture Notes for Part II of the Experimental and Theoretical Physics Tripos, University of Cambridge* (unpublished)

[11] See, for example, S. Gasiorowicz *Quantum Physics, Third Edition* (Wiley, 2003)

[12] Osakabe et.al. *Experimental confirmation of Aharonov-Bohm effect using a toroidal magnetic field confined by a superconductor* Phys. Rev. A 34, 815 - 822 (1986)

[13] L. Moriconi *An elementary derivation of the harmonic oscillator propagator* Am. J. Phys. 72, 1258 (2004), DOI:10.1119/1.1715108

[14] The reader should be warned that the use of the propagator in classical mechanics is not standard in the literature. We feel, however, that it is pedagogically advantageous in the context of this paper.

[15] Although our focus in this paper is on mechanical systems, the approach is far more powerful and can be generalised to deal with fields and other more exotic entities [7].