Simple Harmonic Motion

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- The physical conditions necessary for a system to undergo simple harmonic motion is for the force on the system to be proportional to its displacement, but in the opposite direction.
- The differential equations satisfied by systems undergoing SHM is

$$\boxed{\ddot{x} = -\omega^2 x}$$

  And the general solution is

$$\boxed{x = A\cos\omega t + B\sin\omega t = C\cos(\omega t + \phi)}$$

  ω is the **angular frequency** of the oscillation, and

$$\boxed{T = \frac{2\pi}{\omega}}$$

- Note that:
  - The **velocity** *leads* the displacement by $\pi/2$.
  - The **acceleration** *leads* the displacement by π.

Common Systems

- For a mass m on a spring with spring constant k

$$\boxed{\omega^2 = \frac{k}{m}}$$

  Note that ω is the same whether the spring is horizontal or vertical. This is because a vertical spring is already extended at $t = 0$, and this provides a constant upwards force to counteract gravity.
- For small oscillations of a pendulum length l

$$\boxed{\omega^2 = \frac{g}{l}}$$

- For a torsional pendulum, in which the wire (torsional fibre) exerts a restoring couple of $G = -\tau\theta$ on a bar of moment of inertia I twisted by an angle θ,

$$\boxed{\omega^2 = \frac{\tau}{I}}$$

Energy in SHM

- It can be shown that all systems subject to a **parabolic potential** also undergo simple harmonic motion. Close to a point of stable equilibrium, this is true for <u>every</u> potential [cf. Taylor Expansion of the potential].
- The average PE and KE of a given system undergoing SHM over a whole number of periods or a long period of time is $\frac{1}{4}ka^2$ (each, where a is the amplitude). This can be deduced, for example, by integrating over a period and dividing by the period. The total energy is constant and shared out in between the two forms, oscillating at a frequency of 2ω.
- Therefore, the average energy of an oscillator is proportional to the square of the amplitude of the oscillation.
- In general, the total energy of a system undergoing SHM is of the form
$$E = \frac{1}{2}\alpha\dot{x}^2 + \frac{1}{2}\beta x^2$$
Assuming energy is conserved ($\Rightarrow \mathrm{d}E/\mathrm{d}T = 0$), we end up with
$$\ddot{x} = -\frac{\beta}{\alpha}x$$
Therefore, we can find ω only from an expression for the total energy.

Superposition of SHMs

- A consequence of the linearity of SHM is that two or more oscillations can be **added** (superimposed) to form another valid oscillation.
- For oscillations of the **same frequency** but of different amplitudes and phases, one can build a phasor diagram and use the cosine rule
$$c^2 = a^2 + b^2 - 2ab\cos C$$
to find the length of the resultant vector.
- For oscillations with **different frequencies** but the same amplitude, the following trigonometric identity helps
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
What perspires is that we get an oscillation with angular frequency $\frac{\omega_1+\omega_2}{2}$, modulated with a frequency $\frac{\omega_1-\omega_2}{2}$.

When the original frequencies are very close, the modulating frequency is very slow. We therefore hear "beats", the frequency of which is $\omega_1 - \omega_2$ (not times ½, because a node occurs *twice* in every cycle).

Complex Representation of SHM

- SHM can be represented by a complex number rather than by sin and cos terms:

$$x = ae^{i(\omega t+\phi)} = Ae^{i\omega t}$$

  Where a is a real number and A is a complex number including phase information.

- Note, when differentiating, that $i = e^{i\pi/2}$, so multiplying by i corresponds to a phase difference of π. Thus, $\dot{x} = ai\omega e^{i(\omega t+\phi)} = a\omega e^{i\left(\omega t+\phi+\frac{\pi}{2}\right)}$.

Damped SHM

- Damping usually introduces a force that is proportional to velocity.
- The equation of motion is then

$$m\ddot{x} + b\dot{x} + kx = 0$$
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

  The characteristic equation is

$$\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0 \Rightarrow x = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

  The general solution is then

$$x = A\exp\left(\left\{-\gamma + \sqrt{\gamma^2 - \omega_0^2}\right\}t\right) + B\exp\left(\left\{-\gamma - \sqrt{\gamma^2 - \omega_0^2}\right\}t\right)$$
$$\boxed{x = e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2-\omega_0^2}} + Be^{-t\sqrt{\gamma^2-\omega_0^2}}\right)}$$

  There are then three possibilities:

  - **Light damping** – if $\boxed{\gamma^2 < \omega_0^2}$ – in such a case, the exponent is complex. We can ensure x stays real by expressing A in complex form and letting $B = A^*$. The resulting solution is

$$x = ae^{-\gamma t}\cos(\omega_1 t + \phi)$$

    Therefore, the only effects of light damping are

    - Decreasing the frequency of oscillations from ω_0 to $\sqrt{\left|\gamma^2 - \omega_0^2\right|}$.

  - Causing the amplitude to decay. The ***decay time*** is defined to be the time taken for the amplitude to decay by a factor of e – this is equal to $1/\gamma$.
- **Heavy damping** – if $\boxed{\gamma^2 > \omega_0^2}$ – in such a case, the exponent is real and we end up with a linear combination of two falling exponentials. The motion is ***aperiodic*** – there is no oscillation, and the displacement can cross the equilibrium point *once* at most.
- **Critical damping** – if $\boxed{\gamma^2 = \omega_0^2}$ – in such a case, the general solution is

$$x = e^{-\gamma t}\left(A + Bt\right)$$

there is at most one turning point, and the curve crosses the equilibrium point once at most. Given the same starting conditions and ω_0, a critically damped oscillator settles down in the shortest possible time. This is often used in systems such as galvanometers or other measuring instruments that need to quickly settle down, or in car suspensions.

- The energy decay in light damping can be studied in more detail
  - The total energy at any given instant, as we saw, is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Therefore

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{x}\left(m\ddot{x} + kx\right)$$

Given that the equation for damped SHM is $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$ and that $\omega_0^2 = k/m$ this becomes

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -m\gamma\dot{x}^2$$

(You can check, using $Power = Force \times Velocity$ that this is indeed the rate of doing work against the frictional force.)

  - The amplitude of the nth and $(n+1)$th maxima are

$$a_n = ae^{-\gamma t} \qquad a_{n+1} = ae^{-\gamma(t+T)} = ae^{-\gamma(t+[2\pi/\omega_1])}$$

Therefore, the decay in amplitude per cycle is

$$\frac{a_{n+1}}{a_n} = e^{-2\pi\gamma/\omega_1} = e^{-\Delta}$$

Where Δ is the **logarithmic decrement**, defined by

$$\Delta \equiv \ln\left(\tfrac{a_n}{a_{n+1}}\right) = \frac{2\pi\gamma}{\omega_1}$$

- Since the energy is proportional to the square of the amplitude, the energy decays by $e^{-2\Delta}$ per cycle.
- We define the **quality factor, *Q*** of an oscillator as

$$Q = \frac{\omega_0}{2\gamma}$$

  A larger Q means a smaller γ and therefore a better (less damped) oscillator.
- For very light damping, $\gamma^2 \ll \omega_0^2$, so $\omega_1^2 = \omega_0^2 - \gamma^2 \approx \omega_0^2$. Under these circumstances, we note that

$$\Delta = \frac{2\pi\gamma}{\omega_1} \approx \frac{2\pi\gamma}{\omega_0} = \frac{\pi}{Q} \Rightarrow Q \approx \frac{\pi}{\Delta}$$

  Therefore, the larger Q and the smaller Δ, the weaker the damping and the better the oscillation.
- We know that the energy decays by $e^{-2\Delta}$ per cycle. Therefore, after N cycles, the energy has decayed by $e^{-2N\Delta}$. Thus, we can find the number of cycles after which the energy has decayed by e^{-1},

$$e^{-2N\Delta} = e^{-1}$$

$$-2N\Delta = -1$$

$$N = \frac{1}{2\Delta} \approx \frac{Q}{2\pi}$$

  This means that $Q \approx 2\pi N$ – in other words, another interpretations of Q is the **number of radians through which the damped system oscillates as its energy drops by a factor of *e***. As expected, large Q means good quality and slow decay.

Tips and Tricks

- Take the origin at the equilibrium position when doing things from first principles – it eliminates g from the equations.
- Remember that the actual formula for the k of a spring is $k = \lambda / l_0$. Therefore, if the length of a spring is halved, its k doubles.