Vector Calculus

Suffix Notation

- We define the **Knonecker Delta** and the **Levi-Civita permutation symbol** as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permuation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permuation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

- We then have that

$$\boldsymbol{a} \times \boldsymbol{b} = \varepsilon_{ijk} a_j b_k$$

$$\det \boldsymbol{A} = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$$

- The general identity

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

  Can be established by the following argument:
  - It is equal to 1 when $(i,j,k) = (l,m,n) = (1,2,3)$ (the matrix, in that case, is simply identity matrix).
  - Changes sign when any of (i,j,k) or (l,m,n) are interchanged (by the rules of determinants).
  - This last property also implies that if any of the (i,j,k) or (l,m,n) are equal, it is equal to 0. [Swapping those two equal indices gives $x = -x$, which gives $x = 0$].

  These properties ensure that the RHS and LHS are equal for any index.
- If we contract the identity once by setting $l = i$, we get

$$\boxed{\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}}$$

This is the most useful form to remember.

Vector Differential Operators

- **GRAD**
  - If we consider a **scalar field**, $\Phi(x,y,z) = \Phi(\boldsymbol{r})$, Taylor's Theorem states that

    $$\Phi(x+\delta x, y+\delta y, z+\delta z) = \Phi(x,y,z) + \frac{\partial \Phi}{\partial x}\delta_x + \frac{\partial \Phi}{\partial y}\delta_y + \frac{\partial \Phi}{\partial z}\delta_z + O(\delta x^2, \delta x \delta y, \ldots)$$

    Or

    $$\Phi(\boldsymbol{r}+\delta\boldsymbol{r}) = \Phi(\boldsymbol{r}) + (\nabla\Phi)\cdot\delta\boldsymbol{r} + O\left(\left|\delta\boldsymbol{r}\right|^2\right)$$

    Where

    $$\nabla = \frac{\partial}{\partial x}\boldsymbol{e}_x + \frac{\partial}{\partial y}\boldsymbol{e}_y + \frac{\partial}{\partial z}\boldsymbol{e}_z$$

    $$\boxed{\nabla = \boldsymbol{e}_i \frac{\partial}{\partial x_i}}$$

    And for an **infinitesimal increment**, we can write

    $$\mathrm{d}\Phi = (\nabla\Phi)\cdot\mathrm{d}\boldsymbol{r}$$

    And the grad operator is $\nabla\Phi$.
  - The geometrical interpretation of the **grad** operator is that

    $$\hat{\boldsymbol{t}}\cdot\nabla\Phi$$

    Is the **directional derivative** – the rate of change of Φ with distance in the direction $\hat{\boldsymbol{t}}$.
  - Note that
    - The derivative is **maximal** in the direction $\boldsymbol{t} \parallel \nabla\Phi$.
    - The derivative is **zero** in directions such that $\boldsymbol{t} \perp \nabla\Phi$. These directions therefore lie in the plane **tangent** to the surface $\Phi = \text{constant}$.

    In other words, $\nabla\Phi$ is in the direction of increase of the grad field.
  - Furthermore

- The unit vector **normal** to the surface $\Phi = \text{constant}$ is then $\boldsymbol{n} = \nabla\Phi / |\nabla\Phi|$.
- The rate of change of Φ with arclength s along a curve is $\boldsymbol{t}\cdot\nabla\Phi$ where $\boldsymbol{t} = \mathrm{d}\boldsymbol{r}/\mathrm{d}s$ is the **unit tangent vector** to the **curve**.

- **OTHER DIFFERENTIAL OPERATORS**
  - The **divergence** of a vector field is the **scalar field**

$$\nabla\cdot\boldsymbol{F} = \left(\boldsymbol{e}_i \frac{\partial}{\partial x_i}\right)\cdot\boldsymbol{F} = \frac{\partial F_i}{\partial x_i}$$

  - The **curl** of a vector field is the **vector field**

$$\nabla\times\boldsymbol{F} = \left(\boldsymbol{e}_i \frac{\partial}{\partial x_i}\right)\cdot\boldsymbol{F} = \boldsymbol{e}_i\varepsilon_{ijk}\frac{\partial F_k}{\partial x_j}$$

  This can also be written as a determinant

$$\nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

  - The **Laplacian** of a **scalar field** is the **scalar field**

$$\nabla^2\Phi = \nabla\cdot(\nabla\Phi) = \frac{\partial^2\Phi}{\partial x_i \partial x_i}$$

  The Laplacian of a **vector field** is

$$\nabla^2\boldsymbol{F} = \boldsymbol{e}_i \frac{\partial^2 F_i}{\partial x_j \partial x_j}$$

- **VECTOR DIFFERENTIAL IDENTITIES**
  - There are a number of identities relating different vector differential operators.
  - **Two operators, one field**

$$\nabla\cdot(\nabla\Phi) = \nabla^2\Phi$$

$$\nabla\cdot(\nabla\times\boldsymbol{F}) = 0$$

$$\nabla\times(\nabla\Phi) = \boldsymbol{0}$$

$$\nabla\times(\nabla\times\boldsymbol{F}) = \nabla(\nabla\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}$$

- **One operator, two fields**

$$\nabla(\Psi\Phi) = \Psi\nabla\Phi + \Phi\nabla\Psi$$
$$\nabla\cdot(\Phi\boldsymbol{F}) = (\nabla\Phi)\cdot\boldsymbol{F} + \Phi\nabla\cdot\boldsymbol{F}$$
$$\nabla\times(\Phi\boldsymbol{F}) = (\nabla\Phi)\times\boldsymbol{F} + \Phi\nabla\times\boldsymbol{F}$$
$$\nabla\cdot(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}\cdot(\nabla\times\boldsymbol{F}) - \boldsymbol{F}\cdot(\nabla\times\boldsymbol{G})$$
$$\nabla\times(\boldsymbol{F}\times\boldsymbol{G}) = (\boldsymbol{G}\cdot\nabla)\boldsymbol{F} - \boldsymbol{G}(\nabla\cdot\boldsymbol{F}) - (\boldsymbol{F}\cdot\nabla)\boldsymbol{G} + \boldsymbol{F}(\nabla\cdot\boldsymbol{G})$$
$$\nabla(\boldsymbol{F}\cdot\boldsymbol{G}) = (\boldsymbol{G}\cdot\nabla)\boldsymbol{F} + \boldsymbol{G}\times(\nabla\times\boldsymbol{F}) + (\boldsymbol{F}\cdot\nabla)\boldsymbol{G} + \boldsymbol{F}\times(\nabla\times\boldsymbol{G})$$

- As a result of some of these identities, we have the interesting fact that:
  - If a vector field $\boldsymbol{F}$ is **irrotational** ($\nabla\times\boldsymbol{F} = \boldsymbol{0}$), it can be written as the **gradient** of a **scalar potential** – $\boldsymbol{F} = \nabla\Phi$.
  - If a vector field $\boldsymbol{F}$ is **solenoidal** ($\nabla\cdot\boldsymbol{F} = 0$), it can be written as the **curl** of a **vector potential** – $\boldsymbol{F} = \nabla\times\boldsymbol{G}$.

Integral Theorems

- The **gradient theorem** states that

$$\int_{r_1\to r_2} (\nabla\Phi)\cdot \mathrm{d}\boldsymbol{r} = \Phi(\boldsymbol{r_2}) - \Phi(\boldsymbol{r_1})$$

- The **divergence theorem** (Gauss' Theorem) states that

$$\int_V (\nabla\cdot\boldsymbol{F})\,\mathrm{d}V = \oint_S \boldsymbol{F}\,\mathrm{d}\boldsymbol{S}$$

Where V is the volume **bounded** by the **closed surface** S, and the vector surface element is $\mathrm{d}\boldsymbol{S} = \boldsymbol{n}\mathrm{d}S$, where $\boldsymbol{n}$ is the outward unit normal vector.

For **multiply connected volumes** (eg: spherical shells), *all* the surfaces must be considered.

Related results are as follows:

$$\int_V (\nabla\Phi)\,\mathrm{d}V = \int_S \Phi\,\mathrm{d}\boldsymbol{S}$$
$$\int_V (\nabla\times\boldsymbol{F})\,\mathrm{d}V = \int_S \mathrm{d}\boldsymbol{S}\times\boldsymbol{F}$$

The rule is, effectively, to replace the ∇ in the volume integral by $\boldsymbol{n}$ in the surface one, and the $\mathrm{d}V$ by a $\mathrm{d}S$.

- The **curl theorem** (Stokes' Theorem) states that

$$\int_S (\nabla \times F) \cdot \mathrm{d}\boldsymbol{S} = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

  Where S is an **open** surface bounded by the **closed curve** C. The direction of $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}\boldsymbol{r}$ are chosen so that they form a right-handed system.

  Again, a multiply connected surface (such as an annulus) maybe have more than one bounding curve.

- We can use these integral theorems to get **geometrical interpretations** of grad, div and curl.
  - Consider the gradient theorem to a tiny line segment $\delta \boldsymbol{r} = \hat{\boldsymbol{t}} \delta s$. Since the variation of Φ and $\hat{\boldsymbol{t}}$ along the line are negligible, we have

$$\boldsymbol{t} \cdot (\nabla \Phi) \delta s \approx \delta \Phi$$

$$\boldsymbol{t} \cdot (\nabla \Phi) = \lim_{\delta s \to 0} \frac{\partial \Phi}{\partial s}$$

    The rate of change with distance.
  - Applying the divergence theorem to an arbitrarily small volume δV bounded by δS:

$$\nabla \cdot \boldsymbol{F} = \lim_{\partial V \to 0} \frac{1}{\delta V} \int_{\delta S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

    The efflux per unit volume.
  - Finally, applying the url theorem to an arbitrarily small open surface δS with a unit normal vector $\boldsymbol{n}$ and bounded by a curve δC, we find:

$$\boldsymbol{n} \cdot (\nabla \times \boldsymbol{F}) = \lim_{\delta S \to 0} \frac{1}{\delta S} \int_{\delta C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

    The circulation per unit area.

Orthogonal Curvilinear Coordinates

- **Cartesian coordinates** can be replaced with any independent set of coordinates

$(q_1[x_1, x_2, x_3], q_2[x_1, x_2, x_3], q_3[x_1, x_2, x_3])$. **Curvilinear** (as opposed to **rectilinear**) means that the coordinates "axes" are curves.

- In general, curvilinear coordinates, the **line element** is given by

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{h_1}\mathrm{d}q_1 + \boldsymbol{h_2}\mathrm{d}q_2 + \boldsymbol{h_3}\mathrm{d}q_3$$

Where:

$$\boldsymbol{h}_i = \boldsymbol{e}_i h_i = \frac{\partial \boldsymbol{r}}{\partial q_i} \qquad \text{(No sum)}$$

This determines the **displacement** associated with an **increment** in q_i. We have

  - h_i is the **scale factor** associated with the coordinate q_i. It converts the coordinate increment (which might be an angle, for example) into a **length**. This depends on position.
  - $\boldsymbol{e_i}$ is the corresponding **unit vector**. In general, this will also depend on positive.

If, at any point, $h_i = 0$, then we have a **coordinate singularity** – however much we change the component q_i, we go nowhere.

- To find the surfaces described by keeping a certain coordinate constant, assume that it is constant and twiddle with the expressions obtained to get something recognisable. (Eliminate all non-Cartesian variables except for the one we want to keep constant). To prove that they are perpendicular, show that (for example) $\partial z / \partial x|_u \, \partial z / \partial x|_v = -1$.
- The **Jacobian** of (x, y, z) with respect to (q_1, q_2, q_3) is defined as:

$$J = \frac{\partial(x, y, z)}{\partial(q_1, q_2, q_3)} = \begin{vmatrix} \partial x / \partial q_1 & \partial x / \partial q_2 & \partial x / \partial q_3 \\ \partial y / \partial q_1 & \partial y / \partial q_2 & \partial y / \partial q_3 \\ \partial z / \partial q_1 & \partial z / \partial q_2 & \partial z / \partial q_3 \end{vmatrix}$$

The **columns** of the Jacobian are the vectors $\boldsymbol{h}_i$ as defined above. Therefore

$$J = \boldsymbol{h}_1 \cdot \boldsymbol{h}_2 \times \boldsymbol{h}_3$$

- The **volume element** in a **general curvilinear coordinate system** is therefore

$$\mathrm{d}V = \left|\frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)}\right| \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3$$

The Jacobian therefore appears whenever changing variables in a multiple integral. The modulus sign appears because when changing the limits of integration, one is likely to place them in the right direction (ie: upper limits greater than lower limits), even if q actually decreases as x increases.

- If we consider three sets of n variables, α_i, β_i and γ_i, then, by the chain rule for partial differentiation:

$$\frac{\partial \alpha_i}{\partial \gamma_j} = \sum_{k=1}^{n} \frac{\partial \alpha_i}{\partial \beta_k}\frac{\partial \beta_k}{\partial \gamma_j}$$

Taking the determinant of this matrix equation, we find that:

$$\frac{\partial(\alpha_1,\cdots,\alpha_n)}{\partial(\gamma_1,\cdots,\gamma_n)} = \frac{\partial(\alpha_1,\cdots,\alpha_n)}{\partial(\beta_1,\cdots,\beta_n)}\frac{\partial(\beta_1,\cdots,\beta_n)}{\partial(\gamma_1,\cdots,\gamma_n)}$$

In other words, the Jacobian of a composite transformation is the product of the Jacobians of the transformations of which it is composed. In the special case where $\gamma_i = \alpha_i$ for all i, we get a rule for the inversion of a Jacobian.

- Things are made easier when the coordinates we choose are orthogonal:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

and right-handed

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$$

In this case

  - The **line element** is given by $\mathrm{d}\boldsymbol{r} = \boldsymbol{e}_1 h_1 \mathrm{d}q_1$
  - The **surface element** is given by $\mathrm{d}\boldsymbol{S} = \boldsymbol{e}_3 h_1 h_2 \mathrm{d}q_1 \mathrm{d}q_2$.

- The **volume element** is given by $\mathrm{d}V = h_1 h_2 h_3 \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3$.
- The Jacobian is simply $J = h_1 h_2 h_3$.

- In general

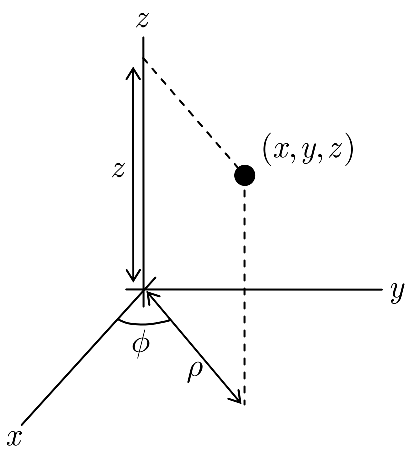
$$\nabla\Phi = \frac{\boldsymbol{e}_1}{h_1}\frac{\partial\Phi}{\partial q_1} + \frac{\boldsymbol{e}_2}{h_2}\frac{\partial\Phi}{\partial q_2} + \frac{\boldsymbol{e}_3}{h_3}\frac{\partial\Phi}{\partial q_3}$$

$$\nabla\cdot\boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 F_1) + \frac{\partial}{\partial q_2}(h_3 h_1 F_2) + \frac{\partial}{\partial q_3}(h_1 h_2 F_3)\right]$$

$$\nabla\times\boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\Phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\Phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\Phi}{\partial q_3}\right)\right]$$

- Commonly used orthogonal coordinate systems are:
  - **Cartesian coordinates**
  - **Cylindrical polar coordinates**

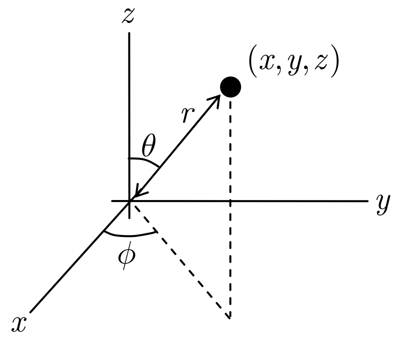


Where:

- $0 < \rho < \infty$, $0 \le \phi < 2\pi$, $-\infty < z < \infty$.
- $\boldsymbol{r} = (x, y, z) = (\rho\cos\phi, \rho\sin\phi, z)$
- $\boldsymbol{h}_\rho = \dfrac{\partial\boldsymbol{r}}{\partial\rho} = (\cos\phi, \sin\phi, 0)$
- $\boldsymbol{h}_\phi = \dfrac{\partial\boldsymbol{r}}{\partial\phi} = (-\rho\sin\phi, \rho\cos\phi, 0)$
- $\boldsymbol{h}_z = \dfrac{\partial\boldsymbol{r}}{\partial z} = (0, 0, 1)$
- $h_\rho = 1 \qquad \boldsymbol{e}_\rho = (\cos\phi, \sin\phi, 0)$

- $h_\phi = \rho$ $\quad\quad$ $\boldsymbol{e}_\phi = (-\sin\phi, \cos\phi, 0)$
- $h_z = 1$ $\quad\quad$ $\boldsymbol{e}_z = (0,0,1)$
- $\mathrm{d}V = \rho \; \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}z$
- The system is **singular** on the axis $\rho = 0$.

- **Spherical polar coordinates**



Where:

- $0 < r < \infty$, $0 < \theta < \pi$, $0 \le \phi < 2\pi$.
- $\boldsymbol{r} = (x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$
- $\boldsymbol{h}_r = \dfrac{\partial \boldsymbol{r}}{\partial r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
- $\boldsymbol{h}_\theta = \dfrac{\partial \boldsymbol{r}}{\partial \theta} = (r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta)$
- $\boldsymbol{h}_\phi = \dfrac{\partial \boldsymbol{r}}{\partial \phi} = (-r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0)$
- $h_r = 1$ $\quad\quad$ $\boldsymbol{e}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
- $h_\theta = r$ $\quad\quad$ $\boldsymbol{e}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$
- $h_\phi = r\sin\theta$ $\quad$ $\boldsymbol{e}_z = (-\sin\phi, \cos\phi, 0)$
- $\mathrm{d}V = r^2 \sin\theta \; \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$
- The system is **singular** on the axis $r = 0$, $\theta = 0$ and $\theta = \pi$.
- Related to cylindrical polars by

$$\rho = r\sin\theta$$

$$z = r\cos\theta$$

