Complex Analysis

Functions of Continuous Variables

- The function ***f(z)*** tends to the limit L as $z \to z_0$ if, for any positive number ε, there exists a positive number δ (depending on ε), such that $|f(z) - L| < \varepsilon$ for all z such that $|z - z_0| < \delta$.
- The function ***f(z)*** is **continuous** at the point $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.
- The function ***f(z)*** is **bounded** as $z \to z_0$ if there exists positive numbers K and δ such that $|f(z)| < K$ for all z with $|z - z_0| < \delta$.
- To find the definitions when $z_0 = \infty$, we simply replace the $|z - z_0| < \delta$ statement in each of those by $|z| > R$.
- O notation:
    - $f(z) = O(g(z))$ as $z \to z_0$ means that $f(z)/g(z)$ is bounded as $z \to z_0$.
    - $f(z) = o(g(z))$ as $z \to z_0$ means that $f(z)/g(z) \to 0$ as $z \to z_0$.
    - $f(z) \sim g(z)$ as $z \to z_0$ means that $f(z)/g(z) \to 1$ as $z \to z_0$. [This means that f is **asymptotically equal** to g – but this shouldn't be written $f(z) \to g(z)$!]
    - Notes:
        - These also apply when $z_0 = \infty$.
        - $f(z) = O(1)$ means that $f(z)$ is bounded.
        - Both the latter of these relations imply the former.
        - $f(z) \sim g(z)$ is a **symmetric relation**.
- **Taylor's Theorem** for functions of a **real** variable states that:

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x_0)+R_n$$

Where

$$R_n=\int_{x_0}^{x_0+h}\frac{(x_0+h-x)^{n-1}}{(n-1)!}f^{(n)}(x)\,\mathrm{d}x$$

Is the remainder after n terms of the Taylor series. (Which can apparently be proved by multiplying R_n by parts, n times). Lagrange's expression for the remainder is:

$$R_n=\frac{h^n}{n!}f^{(n)}(\xi)$$

Where ξ is unknown, in the interval $x_0<\xi<x_0+h$. Therefore:

$$R_n=O(h^n)$$

If *f(x)* is infinitely differentiable in the interval, then it is a **smooth** function, and we can write an infinite Taylor Series:

$$f(x_0+h)=\sum_{n=0}^{\infty}\frac{h^n}{n!}f^{(n)}(x_0)$$

Functions of Complex Variables

- The **derivative** of a function *f(z)* at the point $z = z_0$ is

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

$$f'(z)=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}$$

Requiring a function of a **complex** variable to be differentiable is a surprisingly strong constraint, because it requires the limit to be the same when $\delta z\to 0$ in *any* direction in the complex plane.

- Consider a function $f(x + iy) = u + iv$. If it is to be differentiable at $f(z)$, then we know that:
  - It must certainly be differentiable as we approach $f(z)$ along the real axis:

$$\begin{aligned} f'(z) &= \lim_{\delta x \to 0} \frac{f(z+\delta x) - f(z)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) + \mathrm{i}v(x+\delta x, y) - u(x,y) - \mathrm{i}v(x,y)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + \lim_{\delta x \to 0} \frac{\mathrm{i}v(x+\delta x, y) - \mathrm{i}v(x,y)}{\delta x} \\ &= \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} \end{aligned}$$

  - But also as we approach z along the imaginary axis:

$$\begin{aligned} f'(z) &= \lim_{\delta y \to 0} \frac{f(z+\mathrm{i}\delta y) - f(z)}{\mathrm{i}\delta y} \\ &= \lim_{\delta y \to 0} \frac{u(x, y+\delta y) + \mathrm{i}v(x, y+\delta y) - u(x,y) - \mathrm{i}v(x,y)}{\mathrm{i}\delta y} \\ &= -\mathrm{i}\lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \lim_{\delta x \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\delta y} \\ &= -\mathrm{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned}$$

  - And these **must** be equal. Therefore, a **required** condition for the function to be differentiable at z is

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

  - It turns out that these are also **sufficient conditions**, provided that the partial derivatives are **continuous**.
- A function that has a **complex derivative** at **every point *z* in a region *R*** of the complex plane is said to be **analytic in *R***. To be analytic at z_0 a function must be differentiable throughout some **neighbourhood** $|z - z_0| < \varepsilon$ of that point.
  - Note that this definition implies that even though the CR equations hold at $x = y = 0$ for

$f(z)=|z|^2$, the function is **not** analytic even there, because there is *no* neighbourhood of 0 $|z|<0$ in which the CR equations hold.

- Examples of functions analytic in the **whole complex plane** (**entire functions**) are:
  - $f(z) = c$
  - $f(z) = z^n$ (n 0 or $+ve$ integer)
  - $f(z)$ = a general polynomial function with complex coefficients.
  - $f(z) = e^z$
- Sums, products and compositions of analytic functions are also analytic, and the usual product, quotient and chain rules apply.
- Many functions are analytic everywhere except at isolated points, called **singular points** or **singularities**. For example:
  - $f(z)=P(z)/Q(z)$ where P and Q are polynomials. This is called a **rational function** and is analytic except at points where $Q(z) = 0$.
  - $f(z) = z^c$ where c is a complex constant, is analytic except at $z = 0$ (unless c is a nonnegative integer).
  - $f(z) = \log z$ is also analytic except at $z = 0$
- Consequences of the CR equations:
  - If we know the complex/real part, we can find the other part up to an additive constant.
  - The real and imaginary parts of an analytic function satisfy Laplace's Equation (they are **harmonic functions**):

$$\nabla^2 u = \nabla^2 v = 0$$

  - Furthermore

$$\nabla u \cdot \nabla v = 0$$

So u and v are said to be **conjugate harmonic functions**.

- If we consider f as a function of z and z^* as opposed to x and y [where $x = \frac{1}{2}(z + z^*)$ and $y = \frac{1}{2\mathrm{i}}(z - z^*)$] we can write:

$$\begin{aligned}\frac{\partial f}{\partial z^*} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*} \\ &= \left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)\left(\frac{1}{2}\right) + \left(\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y}\right)\left(-\frac{1}{2i}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\end{aligned}$$

Now, if the function is analytic and the CR are true, then we must have that

$$\frac{\partial f}{\partial z^*} \equiv 0$$

Which shows us that analytic functions <u>cannot</u> be a function of z^*. They can include x and y only in the combination $x + \mathrm{i}y$, but not $x - \mathrm{i}y$.

- If a function of a complex variable is analytic in a region R of the complex plane, not only is it differentiable everywhere in R, it is also differentiable any number of times. If *f(z)* is analytic at $z = z_0$, it has an infinite Taylor Series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

which converges within some neighbourhood of z_0. In fact, this can be taken as a definition of analyticity.

Zeros, Poles and Essential Singularities

- The zeros of *f(z)* are the points $z = z_0$ in the complex plane where $f(z_0) = 0$. A zero is of **order *N*** if

$$f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0 \quad but \quad f^{(N)}(z_0) \neq 0$$

Indeed, if a function f has an N^{th} order 0, then

$$f(z) \sim a_N (z - z_0)^N \qquad\qquad z \to z_0$$

A zero of order 0 is a **simple zero**, one of order 2 is a **double zero**, etc…

If we look at a surface plot, then the higher order the zero, the more complex/angular the geometry is about the zero.

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