Elementary Analysis – Series

- Given a sequence of values $\{u_n\}$ (where $n = 0, 1, 2, \ldots$), we define $S_N = \sum_{n=0}^{N} u_n$.

> The infinite series $S_N = \sum_{n=0}^{N} u_n$ is said to converge to S
>
> *If and only if*
>
> $\lim_{N\to\infty} S_N$ exists and is equal to S. A series that does not converge diverges.

Notes:

- Adding or removing a finite number of terms to the series does not alter its convergence.
- If the series converges, then $\lim_{n\to\infty} u_n = 0$. The converse is not necessarily true – ie: this is a necessary but not sufficient condition for convergence.
- If, however, $\lim_{n\to\infty} u_n \neq 0$, then we can state with certainty that the series diverges.
- The u_n terms may be complex, in which case we require both the complex and imaginary parts to converge for convergence.
- It makes no sense to do arithmetic with divergent series. With convergent series, however, we can say that:
  - If $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$, then $\sum_{n=0}^{\infty} a_n + b_n = A + B$.
  - If $\sum_{n=0}^{\infty} a_n = A$, then $\sum_{n=0}^{\infty} \lambda a_n = \lambda A$, for $\lambda \in \mathbb{R}$.
  - Term-by-term differentiation or integration will not necessarily lead to a new series with the same convergence properties.

- In **arithmetic series**, the difference between successive terms is constant:

> The general term of an arithmetic series is given by $u_n = a + (n-1)d$ $(n = 1, 2, \ldots)$, where a is the first term and d is the common difference. The sum of the series to N terms is
>
> $$S_N = \sum_{n=0}^{N-1} a + nd = \tfrac{1}{2} N \left[2a + (n-1)d\right] = \tfrac{1}{2} N \left[u_1 + u_N\right]$$
>
> The series clearly diverges.

© *Daniel Guetta, 2006*

- In **geometric series**, the ratio of successive terms is constant:

> The general term of a geometric series is given by $u_n = ar^{n-1}$ (n = *1, 2, …*), where a is the first term and d is the common ratio. The sum of the series to N terms is
>
> $$S_N = \sum_{n=1}^{N} ar^n = \frac{a\left(1-r^N\right)}{1-r}$$
>
> The series converges for $|r| < 1$ and diverges otherwise.

- In **arithmetico-geometric series**, we have a combination of both:

> The general term of an arithmetico-geometric series is given by $u_n = \left[a+(n-1)d\right]r^{n-1}$ (n = *1, 2, …*). Its sum can be found in a similar way to that of a pure geometric series.

- Some series can be summed using the **difference method**; attempt to write out the general term u_n in terms of a function *f(n)* and *f(n – m)* for various values of m, and write out the sum – most terms cancel out. This allows us to obtain the following useful sums:
  - $\sum_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$ (using $f(n) = n(n+1)(2n+1)$)
  - $\sum_{n=1}^{N} n^3 = \left(\sum_{n=1}^{N} n\right)^2 = \frac{1}{4}N^2(N+1)^2$ (using $f(n) = \left(n(n+1)\right)^2$)
- Tips and tricks for finding sums:
  - It often helps to integrate or differentiate a series. If the series is not already in terms of a variable, then define *f(x)* such that at some x (usually 1), the value of the function is equal to the sum of the series (see RHB, pp 123).
  - An appropriate substitution can often help – for example, using exponentials in a series with terms that contain trigonometric functions.
- Rules for convergence of infinite series with positive terms
  - If $\lim_{n\to\infty} u_n \neq 0$, the series **diverges**.
  - **Comparison test** [useful for series with dominant term n^p, and for series involving trigonometric terms] – if $\sum_{n=0}^{\infty} b_n$ converges and $a_n \leq b_n$ for all n larger than a given N (ie: eventually), then $\sum_{n=0}^{\infty} a_n$ converges.

- **Ratio test** [useful for series including *n!* or c^n] – we consider a series $\sum_{n=0}^{\infty} a_n$ and find $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Now:
    - If $\rho < 1$, then $\sum_{n=0}^{\infty} a_n$ converges.
    - If $\rho > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.
    - If $\rho = 1$, then this test is inconclusive.
- **Ratio comparison test** – if $\sum_{n=0}^{\infty} b_n$ converges, then if $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$, $\sum_{n=0}^{\infty} a_n$ also converges.
- **Quotient test** [useful in the same situations as the comparison test] – let use consider two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, and say that $\rho = \lim_{n\to\infty} \frac{a_n}{b_n}$. Then:
    - If $\rho \neq 0$ and is finite, then both series behave in the same way.
    - If $\rho = 0$ and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
    - If $\rho = \infty$ and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
- **Integral test** (Maclaurin-Cauchy test) – suppose that there exists a function *f(x)* which monotonically decreases for x greater than some fixed x_0 and for which $f(n) = a_n$ (ie: the value of the function at integer values of x is equal to the corresponding term in the series). Then, if the limit
$$\lim_{N\to\infty} \int^{N} f(x)\,\mathrm{d}x$$
exists, the series $\sum_{n=0}^{\infty} a_n$ is convergent. Otherwise, it is divergent.
- **Cauchy's Root Test** – If we say $\rho = \lim_{n\to\infty} \sqrt[n]{u_n}$, then:
    - If $\rho < 1$, the series converges.
    - If $\rho > 1$, the series diverges.
    - If $\rho = 1$, the test is inconclusive.

• Rules for convergence of infinite series with positive and negative terms
- If $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ also converges, and is said to be absolutely convergent. The terms may be reordered without affecting the convergence of the series.

  To test for absolute convergence, simply apply one of the tests above to $\sum_{n=0}^{\infty} |a_n|$, which will now contain positive terms only.

- If $\sum_{n=0}^{\infty} |a_n|$ does <u>not</u> converge but $\sum_{n=0}^{\infty} a_n$ does, then $\sum_{n=0}^{\infty} a_n$ is said to be <u>conditionally convergent</u>, and the order of the terms can affect the behaviour of the sum.

  For example, in an <u>alternating series</u>, the positive and negative signs alternate. The general form of such a series is:

  $$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

  It turns out that alternating series are convergent if and only if a_n is <u>decreasing</u> and <u>tends to 0</u>.

 © Daniel Guetta, 2006