Elementary Analysis – Limits

- The definition of a limit:

> We say that $\lim_{x \to x_0} f(x) = \kappa$ (or $f(x) \to \kappa$ as $x \to x_0$)
>
> *If and only if*
>
> Given any $\varepsilon > 0$, there exists a δ such that $|f(x) - \kappa| < \varepsilon$ whenever $|x - x_0| < \delta$ and $x \neq x_0$.

Notes:

- In general, δ will depend both on the form of *f(x)* and on the particular value of ε (in other words, δ might be different for different ε).
- In other words, the definition says that *f(x)* tends to κ as long as we can get *f(x)* as close to κ as we want by making x as close to x_0 as we want.
- A limit may exist, not exist or tend to $\pm\infty$.
- The limit can exist even if *f(x)* is not defined at $x = x_0$.
- Limits can also be from above and from below.
- For limits to infinity, the definition becomes:

> We say that $\lim_{x \to \infty} f(x) = \kappa$ (or $f(x) \to \kappa$ as $x \to \infty$)
>
> *If and only if*
>
> Given any $\varepsilon > 0$, there exists an X such that $|f(x) - \kappa| < \varepsilon$ whenever $x > X$.

- Tips and tricks for taking limits:
  - $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
  - $\lim_{x \to a} (f(x) \times g(x)) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$
  - $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ as long as the denominator is *not* 0.
  - When finding limits involving exponents of x, taking logarithms of the required limit is often useful.
- In using the above, it should be noted that:
  - $\pm q \times \infty = \pm\infty$ as long as $q \neq 0$.
  - $\infty + q = \infty$ as long as $q \neq -\infty$.
  - $\frac{q}{\infty} = 0$ as long as $q \neq \pm\infty$.
- For other expressions involving ∞, we can use **L'Hopital's rule**:

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \lim_{x \to a} \left(\frac{f'(x)}{g'(x)} \right)$$

As long as:

- The latter limit exists.
- **EITHER** $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ **OR** $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$.

Other indeterminate forms involving ∞ can be converted to either 0/0 or ∞/∞ using the following table:

	$\lim_{x\to a} f(x)$	$\lim_{x\to a} g(x)$	Converting to 0/0	Converting to ∞/∞
0/0	0	0	N/A	$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{1/g(x)}{1/f(x)}\right)$
∞/∞	∞	∞	$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{1/g(x)}{1/f(x)}\right)$	N/A
$0\times\infty$	0	∞	$\lim_{x\to a}(f(x)\times g(x)) = \lim_{x\to a}\left(\frac{f(x)}{1/g(x)}\right)$	$\lim_{x\to a}(f(x)\times g(x)) = \lim_{x\to a}\left(\frac{g(x)}{1/f(x)}\right)$
$\infty-\infty$	∞	∞	$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a}\left(\frac{1/g(x)-1/f(x)}{1/f(x)g(x)}\right)$	$\lim_{x\to a}(f(x)-g(x)) = \ln\lim_{x\to a}\left(\frac{e^{f(x)}}{e^{g(x)}}\right)$

The rule for the $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ case can easily be proved by expanding the top and bottom of the fraction as a series. The $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ case can be inferred by writing

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{1/g(x)}{1/f(x)}\right)$$

Which is now in the form 0/0. Application of L'Hopital's rule leads to the expected result.

- Strategies for finding limits:
  - Divide by the dominant term. This hopefully gets us terms that tend to 0 at the top.
  - Try and find normally using the rules for limits or l'Hopital's Rule.
  - If all else fails, use a series expansion.
- Note that **factorials** ($n!$) *dominate* **exponentials** (a^n) which *dominate* **powers** (n^b) which *dominate* **logarithms** ($\log n$).
- For any limits to 0, give the direction the 0 is approached.
- If $f(x) \sim x^n$ as $x \to a$, then we say that *f(x)* is $O(x^n)$ as $x \to a$

> We say that *f(x)* is $O(x^n)$ as $x \to \infty$
>
> *If and only if*
>
> There exist X and κ such that $\left|\frac{f(x)}{x^n}\right| < \kappa$ for all $x > X$.