Differential Equations

Introduction

- Many physical situations can be described by **differential equations**. The general form of an n^{th} order differential equation is

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = F\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}, \frac{\mathrm{d}^{n-2} y}{\mathrm{d}x^{n-2}}, \cdots, \frac{\mathrm{d}y}{\mathrm{d}x}, x, y\right)$$

First Order Differential Equations

- The general form of a first order differential equation is $\mathrm{d}y / \mathrm{d}x = F(x,y)$. In general it is only possible to write down a closed-form solution of this equation for certain forms of *F(x, y)*.
- When solving these, look for the following, in order:
  - Factorise and simplify as much as possible.
  - Separatable equations – ie: those of the form *F(x, y) = g(x)f(y)*.
  - Linear equations, of the form

    $$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$$

    In such a case, simply multiple both sides by the integrating factor $I(x) = e^{\int p(x)\,\mathrm{d}x}$. The equation then becomes:

    $$\frac{\mathrm{d}}{\mathrm{d}x}\left[I(x)y\right] = I(x)q(x)$$

    Which is hopefully nice and solvable.
  - Try substitutions:
    - **Homogeneous equations**, of the form

      $$\frac{\mathrm{d}y}{\mathrm{d}x} = H\left(\frac{x}{y}\right)$$

      In other words, equations in which stretching the *(x, y)* plane in both directions makes no difference, use the substitution

      $$y = ux$$

      The best way to identify a homogeneous equation is to notice that the only bits in the equation that include x and y are of the form *x/y*. If something looks a bit complicated, just divide by an appropriate power of x and y throughout and see if it becomes as required. [In general, we require the sum of powers of x and y in each of the terms at the top and bottom of the fraction to be the same].

- **Isobaric equations** are a generalisation of homogenous ones. We write the equation out in terms of differentials, and then give y and dy a weight of m and x and dx a weight of 1 to make everything dimensionally consistent. Then, substitute $y = vx^m$.
- **Bernouli style equations**, of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

The key step here is to divide by some power of y – usually one less power than n, to get:

$$\frac{1}{y^{n-1}}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\frac{1}{y^{n-2}} = Q(x)y$$

And then to realise that the two powers of y on the left-hand side are derivatives of one another. This leads to an obvious substitution of $u = 1/y^{n-1}$, and everything happens from there.
- In equations where x and y only appear in the form $ax + by + c$, just make the substitution $u = ax + by + c$.
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- If all else fails, the following are always good substitutions to try:
  - The bit before the dy/dx
  - Any bit bothering us
- An option, to be fancy, is to write it in terms of differentials (see later).

Second Order Differential Equations

- For linear second order differential equations with constant coefficients of the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a\frac{\mathrm{d}y}{\mathrm{d}x} + by = 0$$

We construct the Auxiliary Quadratic:

$$\lambda^2 + a\lambda + b = 0$$

We then get one of the following solutions

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

$$y = e^{\lambda x}(A + Bx)$$

$$y = e^{\alpha t}(A\cos[\beta x] + B\sin[\beta x])$$

- To find a particular integral:
  - If we have a polynomial, construct a polynomial with all powers up to that one.
  - If we have a trig term, construct an expression with both trig terms.

- In all cases, if any of the items already exists in the complementary function, multiply by x repeatedly.

