Calculus

First of all, the usual rubbish:

$\int \frac{\mathrm{d}}{\mathrm{d}x} f(x)$	$\frac{\mathrm{d}}{\mathrm{d}x} f(x)$
$\tan x$	$\sec^2(x)$
$\cot x$	$-\mathrm{cosec}^2(x)$
$\sec x$	$\ln(\sec x + \tan x)$
a^x	$a^x \ln(a)$
$\frac{1}{a}\,\mathrm{atan}\left(\frac{x}{a}\right)$	$\frac{1}{x^2+a^2}$
$\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) = -\frac{1}{a}\,\mathrm{acoth}\left(\frac{x}{a}\right)$	$\frac{1}{x^2-a^2}$ $\left(x^2 > a^2\right)$
$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) = \frac{1}{a}\,\mathrm{atanh}\left(\frac{x}{a}\right)$	$\frac{1}{a^2-x^2}$ $\left(x^2 < a^2\right)$
$\mathrm{asin}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\mathrm{asinh}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2+a^2}}$
$\mathrm{acosh}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2-a^2}}$

Strategies…

Function of	Strategy	
[These also work if the top is a function of ax + b]		
$a^2 - x^2$ $\sqrt{a^2 - x^2}$	Substitute $x = a\sin\theta$	
$a^2 + x^2$ $\sqrt{a^2 + x^2}$	Substitute $x = a\tan\theta$ Substitute $x = a\sinh\theta$	
$x^2 - a^2$ $\sqrt{x^2 - a^2}$	Substitute $x = a\cosh\theta$	
$\mathrm{sech}\ x$	Write in terms of exponentials and substitute $u = e^x$	
Rational function of $\sin x$ and/or $\cos x$	Substitute $t = \tan\frac{x}{2}$, and then use the results $\sin x = \dfrac{2t}{1+t^2}$ $\quad\cos x = \dfrac{1-t^2}{1+t^2}$ $\quad\mathrm{d}x = \dfrac{2}{1+t^2}\,\mathrm{d}t$	
$\sqrt{1-\cos x}$	Use the half-angle formulae to remove the root	
$\sin^m x\cos^n x$	If m is odd	Hold one factor of sin in reserve, and changes all the other sines to cosines. Then, substitute $x = \cos\theta$
	If n is odd	Similar.
	If neither are odd	Use the half-angle formulae to reduce the powers
	OR – expand into lots of $\sin nx$ using complex numbers!	

$\frac{1}{(ax+b)\sqrt{px^2+qx+r}}$	Substitute $ax+b=\frac{1}{u}$
$e^{ax}\cos x$	Convert $\cos x$ into the exponential form of a complex number
$\frac{x^2}{1+x^2}$	Re-write it as $\frac{-1+1+x^2}{1+x^2}$

Partial fractions:

Fraction	Decompose into
$\frac{f(x)}{(x+\alpha)(x+\beta)}$	$\frac{A}{x+\alpha}+\frac{B}{x+\beta}$
$\frac{f(x)}{\left(x^2+\alpha\right)(x+\beta)}$	$\frac{Ax+B}{x^2+\alpha}+\frac{C}{x+\beta}$
$\frac{f(x)}{(x+\alpha)(x+\beta)^n}$	$\frac{A}{x+\alpha}+\frac{B}{x+\beta}+\frac{C}{(x+\beta)^2}+\cdots+\frac{X}{(x+\beta)^n}$

- To prove the product rule:
  - $y(x)=f(x)g(x)$
  - $y(x+\delta x)=f(x+\delta x)g(x+\delta x)=[f(x)+\delta f][g(x)+\delta g]=f(x)g(x)+g(x)\delta f+f(x)\delta g+\delta g\delta f$
  - $\frac{\delta y}{\delta x}=\frac{y(x+\delta x)-y(x)}{\delta x}=\frac{\delta f}{\delta x}g(x)+\frac{\delta g}{\delta x}f(x)+\frac{\delta g\delta f}{\delta x}$
  - $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{\delta f}{\delta x}g(x)+\lim_{\delta x\to 0}\frac{\delta g}{\delta x}f(x)+\lim_{\delta x\to 0}\frac{\delta g\delta f}{\delta x}=\frac{df}{dx}g(x)+\frac{dg}{dx}f(x)$
- To prove the chain rule, let δg be the fluctuation in *g(x)* as x increases by δx

$$\frac{\delta y}{\delta x}=\frac{\delta f}{\delta x}=\lim_{\delta x\to 0}\frac{\delta f}{\delta x}=\lim_{\delta x\to 0}\left(\frac{\delta f}{\delta g}\frac{\delta g}{\delta x}\right)=\lim_{\delta x\to 0}\frac{\delta f}{\delta g}\times\lim_{\delta x\to 0}\frac{\delta g}{\delta x}=\frac{df}{dg}\times\frac{dg}{dx}$$

- To prove integration by parts works:

$$\frac{d}{dx}(fg)=\frac{df}{dx}g+f\frac{dg}{dx}$$

$$f\frac{dg}{dx}=\frac{d}{dx}(fg)-\frac{df}{dx}g$$

$$\int fg'\,\mathrm{d}x=fg-\int f'g\,\mathrm{d}x+C$$

- Leibnitz's Formula is that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(fg)=\sum_{k=0}^{n}\left[{}^nC_k\times\frac{\mathrm{d}^k}{\mathrm{d}x^k}f\times\frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}}g\right]$$

  To prove it:

  - Assume true for N: $f^{(N)}=\sum_{k=0}^{N}{}^NC_kf^{(k)}g^{(N-k)}$
  - Differentiate with respect to x:

$$\frac{\mathrm{d}}{\mathrm{d}x}f^{(N)}=\sum_{k=0}^{N}{}^NC_k\frac{\mathrm{d}}{\mathrm{d}x}\left[f^{(k)}g^{(N-k)}\right]$$

$$f^{(N+1)}=\sum_{k=0}^{N}{}^NC_k\left[f^{(k)}g^{(N-k+1)}+f^{(k+1)}g^{(N-k)}\right]$$

$$=\sum_{s=0}^{N}{}^NC_sf^{(s)}g^{(N-s+1)}+\sum_{s=1}^{N+1}{}^NC_{s-1}f^{(s)}g^{(N-s+1)}$$

- Separate out the first term of the first series and the last term of the other series:

$$f^{(N+1)} = \left({}^{N}C_0 f g^{(N+1)}\right) + \sum_{s=1}^{N} {}^{N}C_s f^{(s)} g^{(N-s+1)} + \sum_{s=1}^{N} {}^{N}C_{s-1} f^{(s)} g^{(N-s+1)} + \left({}^{N}C_N f^{(N+1)} g\right)$$
$$= \left(f g^{(N+1)} + f^{(N+1)} g\right) + \sum_{s=1}^{N} \left({}^{N}C_s + {}^{N}C_{s-1}\right)\left(f^{(s)} g^{(N-s+1)}\right)$$

- Realise that ${}^{N}C_s + {}^{N}C_{s-1} = {}^{N+1}C_s$ as follows:

$$\begin{aligned} {}^{N}C_s + {}^{N}C_{s-1} &= \frac{N!}{s!(N-s)!} + \frac{N!}{(s-1)!(N-s+1)!} \\ &= \frac{(N-s+1)N! + sN!}{s!(N-s+1)!} \\ &= \frac{(N+1)N!}{s!(N-s+1)!} = \frac{(N+1)!}{s!(N+1-s)!} = {}^{N+1}C_s \end{aligned}$$

- Simply feed it in

$$f^{(N+1)} = \left(f g^{(N+1)} + f^{(N+1)} g\right) + \sum_{s=1}^{N} \left({}^{N+1}C_s\right)\left(f^{(s)} g^{(N-s+1)}\right)$$

- Realise that the first two terms are simply the thing inside the summation at $s = 0$ and $s = N + 1$, and that therefore:

$$f^{(N+1)} = \sum_{s=0}^{N+1} {}^{N+1}C_s f^{(s)} g^{(N-s+1)}$$

QED.

- Special points of a function:
  - If $\frac{\mathrm{d}f}{\mathrm{d}x} = 0$, the point is a **stationary point**. In such a case:
    - If $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} > 0$, the point is a **minimum**.
    - If $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} < 0$, the point is a **maximum**.
    - Maxima and minima are also called **turning points**.
  - If $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0$, we have a **point of inflexion** (whatever the value of $\mathrm{d}f/\mathrm{d}x$).
    - There will always be a point of inflexion between a maximum and a minimum.
    - If $\mathrm{d}f/\mathrm{d}x$ is also equal to 0, and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ **changes sign** through the point, then we have a **stationary point of inflection**.
- Graph plotting:
  - Find values as x gets very big and small, values at $x = 0$, and derivatives of all these things.
  - Mark the roots on the graph.
  - Somehow show the envelope, if there is one.

- For graphs like $e^x \cos x$, make sure the period stays constant, and note that maxima and minima aren't as expected!

- Important simplifications:

A function is **EVEN** if $f(x) = f(-x)$
A function is **ODD** if $f(x) = -f(-x)$

- For an **EVEN** function, $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$.
- For an **ODD** function, $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$.
- $\int_{\substack{\text{whole number}\\ \text{of periods}}} \sin^n(x) = \int_{\substack{\text{whole number}\\ \text{of periods}}} \cos^n(x) = 0$ as long as n is an **ODD NUMBER**.

- Stirling's Formula:
  - First, we note that $\ln(n!) = \sum_{x=1}^{n} \ln(x) \sim \int_{1}^{n} \ln(x)\,\mathrm{d}x$ for large n. Now, $\int_{1}^{n} \ln(x)\,\mathrm{d}x = n\ln n - n + 1 \sim n\ln n - n$ for large n. This gives $\ln(n!) \approx n\ln n - n$ and $n! = n^n e^{-n}$. Sadly, however, the latter is a **bad** approximation, because a **small eror** in ln n! leads to a factor in n!
  - We note that the function $\Gamma(n) = \int_{0}^{\infty} x^{n-1} e^{-x}\,\mathrm{d}x$ satisfies $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(1) = 1$. We therefore define $n! = \Gamma(n)$.
  - Now, if we let $x = n + y$, then

$$\ln x = \ln\left(n\left[1+\tfrac{y}{n}\right]\right) = \ln n + \ln\left[1+\tfrac{y}{n}\right] = \ln n + \frac{y}{n} - \frac{y^2}{2n^2} + \frac{y^3}{3n^3} + \cdots$$

  - Now, we can express $\Gamma(n+1)$ as $\Gamma(n+1) = n! = \int_{0}^{\infty} e^{n\ln x - x}\,\mathrm{d}x$. Feeding the expression we obtained for ln x into this, and integrating dy:

$$n! = \int_{-n}^{\infty} \exp\left[n\left(\ln n + \frac{y}{n} - \frac{y^2}{2n^2} + \frac{y^3}{3n^3} + \cdots\right) - (n+y)\right]\mathrm{d}y$$

  If n is sufficiently large, this is

$$\begin{aligned} n! &= \int_{-\infty}^{\infty} \exp\left[n\left(\ln n + \frac{y}{n} - \frac{y^2}{2n^2}\right) - n - y\right]\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \exp\left[n\ln n + y - \frac{y^2}{2n} - n - y\right]\mathrm{d}y \\ &= \int_{-\infty}^{\infty} e^{n\ln n - n} e^{-y^2/2n}\,\mathrm{d}y \\ &= e^{n\ln n - n}\int_{-\infty}^{\infty} e^{-y^2/2n}\,\mathrm{d}y = n^n e^{-n}\sqrt{2\pi n} \end{aligned}$$

- Differentiation of integrals:
  - An integral $\int_{a}^{b} f(x)\,\mathrm{d}x$ is a function of a and b. We can therefore differentiate it with respect to either these two variables:

- $\frac{\partial}{\partial b}\int_a^b f(x)\,\mathrm{d}x = f(b)$ (since increasing the upper limit by δb will **increase** the area by $f(b)\delta b$).
- $\frac{\partial}{\partial a}\int_a^b f(x)\,\mathrm{d}x = -f(a)$ (by swapping limits, or realising that **increasing** the **lower** limit by δa will **decrease** the area by $f(a)\delta a$).

o An integral $\int_a^b f(x,\lambda)\,\mathrm{d}x$ is a function of a and b and λ. We can therefore differentiate it with respect to the parameter. It turns out that $\frac{\partial}{\partial \lambda}\int_a^b f(x,\lambda)\,\mathrm{d}x = \int_a^b \frac{\partial}{\partial \lambda} f(x,\lambda)\,\mathrm{d}x$. This can be rationalised in one of two ways:
  - By noting that changing the parameter by $\delta\lambda$ will change the function *everywhere* in between the two limits, by an amount $\frac{\partial}{\partial \lambda} f(x,\lambda)$. We want the sum of all these changes over the function.
  - By using the definition of an integral as the limit of a sum:
$$\frac{\partial}{\partial \lambda}\int_a^b f(x,\lambda)\,\mathrm{d}x = \frac{\partial}{\partial \lambda}\sum f(x,\lambda)\,\delta x = \sum \frac{\partial}{\partial \lambda} f(x,\lambda)\,\delta x = \int_a^b \frac{\partial}{\partial \lambda} f(x,\lambda)\,\mathrm{d}x$$

o If a certain variable turns up both in the limits and as a parameter, then just add contributions, ignoring the "corners".

o This allows us to do a rather tricky integral. First, note that
$$\int_0^\infty e^{-ax}\,\mathrm{d}x = \left[-\tfrac{1}{a}e^{-ax}\right]_0^\infty = \tfrac{1}{a}$$
We now differentiate this with respect to the parameter a:
$$\frac{\partial}{\partial a}\int_0^\infty e^{-ax}\,\mathrm{d}x = \int_0^\infty \frac{\partial}{\partial a} e^{-ax}\,\mathrm{d}x = \int_0^\infty -xe^{-ax}\,\mathrm{d}x = \frac{\partial}{\partial a}\left(\frac{1}{a}\right) = -\frac{1}{a^2}$$
Doing this repeatedly, we find that
$$\int_0^\infty x^n e^{-ax}\,\mathrm{d}x = \frac{n!}{a^{n+1}}$$
Rather quicker than integration by parts! *Note that if $a = 1$*, this is the Γ function.

- **Schwatz's Inequality** is an extension of the **triangle rule** to **inner products** (generalisations of dot products):
$$\left(\int_a^b fg\,\mathrm{d}x\right)^2 \le \int_a^b f^2\,\mathrm{d}x \int_a^b g^2\,\mathrm{d}x$$
To prove it:

o We know that $\int_a^b (f+\lambda g)^2\,\mathrm{d}x \ge 0$, because the function is positive everywhere. So $\int_a^b f^2\,\mathrm{d}x + 2\lambda\int_a^b fg\,\mathrm{d}x + \lambda^2\int_a^b g^2\,\mathrm{d}x \ge 0$.

o We can assume that $\int_a^b f^2\,\mathrm{d}x \ne 0$ - otherwise, the both sides of the inequality above are 0 and it's trivially true.

- So, doing a bit of re-arrangement:

$$1+2\lambda\frac{\int_a^b fg\,\mathrm{d}x}{\int_a^b f^2\,\mathrm{d}x}+\lambda^2\frac{\int_a^b g^2\,\mathrm{d}x}{\int_a^b f^2\,\mathrm{d}x}\geq 0$$

$$\left(1+\lambda\frac{\int_a^b fg\ \mathrm{d}x}{\int_a^b f^2\ \mathrm{d}x}\right)^2-\lambda^2\left(\frac{\int_a^b fg\ \mathrm{d}x}{\int_a^b f^2\ \mathrm{d}x}\right)^2+\lambda^2\frac{\int_a^b g^2\,\mathrm{d}x}{\int_a^b f^2\,\mathrm{d}x}\geq 0$$

We now chose λ so that the first bracket becomes 0 (this involves dividing by $\int fg$. If this is 0, Schwartz's is trivially true):

$$-\lambda^2\left(\frac{\int_a^b fg\ \mathrm{d}x}{\int_a^b f^2\ \mathrm{d}x}\right)^2+\lambda^2\frac{\int_a^b g^2\,\mathrm{d}x}{\int_a^b f^2\,\mathrm{d}x}\geq 0$$

Since $\lambda \geq 0$, we can cancel and rearrange:

$$\frac{\int_a^b g^2\,\mathrm{d}x}{\int_a^b f^2\,\mathrm{d}x}\geq\left(\frac{\int_a^b fg\ \mathrm{d}x}{\int_a^b f^2\ \mathrm{d}x}\right)^2$$

$$\left(\int_a^b fg\ \mathrm{d}x\right)^2\leq\frac{\int_a^b g^2\,\mathrm{d}x}{\int_a^b f^2\,\mathrm{d}x}\left(\int_a^b f^2\ \mathrm{d}x\right)^2$$

$$\left(\int_a^b fg\ \mathrm{d}x\right)^2\leq\int_a^b g^2\,\mathrm{d}x\int_a^b f^2\,\mathrm{d}x$$

- When doing **simple multiple integrals** in which the limits do **not** depend on each other *and* the function can be separated into an x and a y component, a **simplification** is possible:

$$\int_c^d\int_a^b g(x)h(y)\ \mathrm{d}x\,\mathrm{d}y=\left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right)$$

- The substitution $c=\cos\theta$ is often very useful in spherical polars.
- We can use an elegant trick to evaluate $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$. We first let $I=\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$ and we note that since x is a dummy variable, we can also say $I=\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y$. So

$$I^2=\int_{-\infty}^{\infty} e^{-y^2}\ \mathrm{d}y\int_{-\infty}^{\infty} e^{-x^2}\ \mathrm{d}x=\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)}\,\mathrm{d}x\,\mathrm{d}y=\int_{\text{2D Plane}} e^{-\left(x^2+y^2\right)}\,\mathrm{d}x\,\mathrm{d}y$$

We can now change to **polar coordinates**:

$$\int_{\text{2D Plane}} e^{-\left(x^2+y^2\right)}\,\mathrm{d}x\,\mathrm{d}y=\int_{r=0}^{\infty}\int_{\phi=0}^{2\pi} e^{-r^2}r\,\mathrm{d}r\,\mathrm{d}\phi=\pi$$

Which means that

$$\int_{-\infty}^{\infty} e^{-x^2}\ \mathrm{d}x=\left|\sqrt{\pi}\right|\ \text{ and }\ \int_0^{\infty} e^{-x^2}\ \mathrm{d}x=\tfrac{1}{2}\left|\sqrt{\pi}\right|$$

Since e^{-x^2} is an <u>even</u> function

Thus, the **normalised normal distribution** is given by

$$P(X=x)=\sqrt{\frac{1}{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\qquad(\text{if } X\sim N(\mu,\sigma))$$

