8.02 – Test 3 Review

Daniel Guetta
8.02, Section L03 – Spring 2009
(Dated: April 30, 2009)

I. SELF-INDUCTANCE

The self-inductance is a really, really simple concept. It's really just a definition. It is given by

$$L = \Phi_B / I$$

Where Φ_B is the flux through a component, and I is the current through it.

It basically just tells you how much flux the thing produces for a given current. The higher L, the higher flux it produces, the lower L, the lesser the flux.

Let's work it out for a solenoid of length radius r, and turns per unit length n. The steps are as follows:

1. Imagine a current I is flowing through the solenoid.

2. Calculate the B field in the solenoid that results from that current.

   From our previous studies, we know that it is given by

   $$B = \mu_0 n I$$

3. Calculate the flux through the component.

   In this case, a field only exists *inside* the solenoid, and the inside of the solenoid has surface are πr^2. Furthermore, the field passes through N coils. So the flux is given by

   $$\Phi_B = N \pi r^2 \mu_0 n I$$

   We can write this in terms of the coil density n and the length ℓ

   $$\Phi_B = n \ell \pi r^2 \mu_0 n I$$

4. Finally, divide this flux by I to find L

   $$\boxed{L = n^2 \ell \pi r^2 \mu_0} \tag{1}$$

II. MAGNETIC ENERGY

You know from your previous studies that you can consider the energy stored in an *electric* field in two ways

- By considering the amount of work done in bringing each charge from its original position to its final position (for example, in a capacitor, this gives $E = \frac{1}{2} C V^2$).
- By considering the energy as being stored in the field itself, and using that fact that electric energy density is given by

  $$\boxed{u_E = \frac{1}{2} \epsilon_0 E^2}$$

The situation is very similar in a magnetic field. We can work out the energy stored in two ways

- By considering the amount of work done with each small increase in current to get to the given configuration.
- By considering the energy stored in the magnetic field itself, and using the fact that the magnetic energy density is given by

  $$\boxed{u_B = \frac{1}{2\mu_0} B^2}$$

Let's find the energy stored in a solenoid of length ℓ and turn-density N through which a current I flows, using both methods.

A. Small increases

Imagine that the we want to increase the current in the solenoid by δI. The resulting *change* in flux is given, via the self-inductance, by

$$\delta \Phi = L \delta I$$

And therefore

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = L \frac{\mathrm{d}I}{\mathrm{d}t}$$

Using Faraday's Law, $\epsilon = \frac{\mathrm{d}\phi}{\mathrm{d}t}$, we find that

$$\epsilon = L \frac{\mathrm{d}I}{\mathrm{d}t}$$

This electromotive force acts *against* our attempts to increase the current, and so it will take work to push against it.

How much work? Well, the *power* in a circuit is given by

$$P = VI$$

And so in this case

$$P = LI\frac{\mathrm{d}I}{\mathrm{d}t}$$

To find the total *energy*, we need to *integrate* this power with respect to time

$$E = \int_{t_i}^{t_f} LI\frac{\mathrm{d}I}{\mathrm{d}t}\ \mathrm{d}t$$

'Cancelling the dt'

$$E = \int_{I_i}^{i_f} LI\ \mathrm{d}I$$

And so

$$E = \frac{1}{2}LI^2$$

Using expression 1 for L, we get

$$\boxed{E = \frac{1}{2}n^2\ell\pi r^2\mu_0 I^2}$$

B. Energy density

Using the energy density is much easier. We simply note that the field in the solenoid is given by

$$B = \mu_0 n I$$

with energy density

$$u_B = \frac{1}{2}\mu_0 n^2 I^2$$

We also note that the field elsewhere is 0.

We then remember that the total *volume* containing this field is $\pi r^2\ell$. Since the field is uniform throughout that domain, we have that

$$E = \pi r^2\ell u_B$$

This gives

$$E = \frac{1}{2}\pi r^2\ell\mu_0 n^2 I^2$$

precisely as above.

Note that if the field had not been uniform, we would have had to integrate, to find

$$E = \iiint u_B\ \mathrm{d}V$$

III. MAXWELL'S EQUATIONS

Here's a brief overview of Maxwell's equations and what they're used for

$$\boxed{\iint_A \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}}$$

Where A is any closed surface, and Q_{enclosed} is the charge enclosed within that surface.

This is *Gauss' Law*. It is used to find the *electric field* caused by *continuous charge distributions* which have some *symmetry*.

$$\boxed{\oint_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{\ell} = \mu_0 I_{\text{enclosed}} + \mu_0\epsilon_0\frac{\partial\Phi_{E,S}}{\partial t}}$$

Where S is any loop, I_{enclosed} is the current passing through that loop and Φ_E, S is the electric flux through that loop.

This is the Ampere-Maxwell law. It has two parts to it

- **Ampere's Law** (the first term above) is used to find the *magnetic field* caused by *currents* which have some *symmetry.*
- **Maxwell's Law** (the second term above) defines the *displacement current.* It indicates a way that *changing* electric fields can cause magnetic field.

You might wonder – how do I know the *sign* of $\partial\Phi_E/\partial t$? Here's a brief guide:

- If the *area* is changing (or the inclination of the area), then you're going to have to think (sorry!) Does that mean the field through it is bigger or smaller?
- If the *field* is changing, then $\partial\Phi_E/\partial t$ will be in the *same* direction as the field if the field is *increasing*, or in the *opposite* direction if it's *decreasing.*

$$\boxed{\epsilon = \oint \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{\ell} = -\frac{\partial\phi_{B,S}}{\partial t}}$$

This is Faraday's Law, and it describes the way *changing* magnetic fields can cause electric fields.

The comments above regarding finding the direction of $\partial\Phi_E/\partial t$ also apply in this case for $\partial\Phi_B/\partial t$

$$\boxed{\iint_A \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{A} = 0}$$

This is simply the statement that there are no magnetic monopoles.

IV. WAVES AND ELECTROMAGNETIC RADIATION

A. Waves

There's only a few things you need to know about waves.

- They all satisfy the *wave equation*

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}}$$

  Where c is the wave's speed.

- A wave of a given form is totally defined by three numbers:
  - What defines the *temporal* form of the wave is its *period* T. Sometimes, the frequency, $\nu = 1/T$ is quoted instead.
  - What defines the *spatial* form of the wave is its *wavelength* λ.
  - What defines the *strength* of the wave is its amplitude, A.

  Once you have these two numbers, you know everything about the wave.

- The spatial and temporal part are linked by the *velocity* of the wave by

$$c = \nu\lambda$$

  This formula follows from the wave equation.

- The wavefunction ψ is generally a function of

$$(\text{something1})x \pm (\text{something2})t$$

  Remember the following simple rule

  If the $\pm$ sign above is a '+' , the wave is travelling *backwards*, in the direction of *negative* x.

  If the $\pm$ sign above is a '−' , the wave is travelling *forward*, in the direction of *positive* x.

It is crucial to note that we have see three quantities to define the wave λ, ν and c. Remember that since you know $c = \nu\lambda$, once you know *two* of those, you can easily find the third. Don't be put off by that in an exam.

B. Sinusoidal waves

Everything we said above applies to sinusoidal waves. The only difference is that we now know how to write ψ, the wavefunction. It takes the form

$$\boxed{\psi = A\sin(kx \pm \omega t)}$$

Clearly, the wave is totally defined by k and ω. How do those link to the two defining quantities we talked about above (A, ν and λ)? Clearly, A means the same here and there. It also turns out that

$$\omega = \frac{2\pi}{T} = 2\pi\nu$$

and

$$k = \frac{2\pi}{\lambda}$$

Together, these give

$$c = \nu\lambda = \frac{\omega}{2\pi}\frac{2\pi}{k} = \frac{\omega}{k}$$

These are very useful relations. They mean that if you know two of c, ν, ω, λ, T, you can find all the others.

C. Electromagnetic waves

It turns out cleverly hidden inside Maxwell's Equations are two *wave equations*. One for E and one for B. So it is possible for electromagnetic waves to exist, with the changing electric field causing a magnetic field, and vice-versa.

Maxwell's Equations, however, impose a requirement on the *amplitude* of each of these two waves, E_0 and B_0. It requires that

$$\frac{E_0}{B_0} = c$$

and

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

Finally, it requires that the magnetic and electric field be *perpendicular* to each other. We will see that these are related to the direction of propagation.

The idea of a plane wave is somewhat complicated, and I'll talk about it in class.

1. *Momentum transfer*

I doubt this will come up in an 8.02 exam, but it turns out that electromagnetic waves carry *momentum* with them. If you think about it, it kind of makes sense – they carry energy, so they should be carrying momentum.

It turns out that the *momentum density per unit area* for a wave is given by

$$\boxed{\boldsymbol{p}_{\text{dens}} = \frac{\boldsymbol{S}}{c^2}}$$

It also turns out that if a momentum $\boldsymbol{p}$ is transferred to a surface at a speed v, the force on the surface is

$$\boxed{\boldsymbol{F} = \boldsymbol{p}v}$$

I'm not going to show that either of these are true, because the derivations are cumbersome. You should at least convince yourselves that it's dimensionally consistent.

Let's see how we can use this. Let's imagine that a wave of pointing vector $\boldsymbol{S}$ is incident upon a conductor of area A, and bounces straight back off it. What is the force on the conductor? We can reason this as follows:

- The momentum carried by the wave per unit area is $\boldsymbol{S}/c^2$. So over our sheet, the momentum carried is $A\boldsymbol{S}/c^2$
- As a result of the wave bouncing on the conductor, it reverses its momentum completely. So it transfers a momentum $2A\boldsymbol{S}/c^2$.
- Using the second formula, we therefore find that the total force on the conductor is $2A\boldsymbol{S}/c$.

2. *Reflection off conducting surface*

I'm not even convinced that this topic is fair play for the exam, so I'll only cover it *extremely* briefly.

When an electromagnetic wave bounces off a perfectly conducting surface, a few points must be borne in mind:

- The direction of travel of the electric and magnetic waves reserve. So if the waves used to be in terms of $kx + \omega t$, they'll now be in terms of $kx - \omega t$.
- The electric field at the surface must be 0 (because the electric field is always 0 at a conductor). This means that the incident and reflected electric waves must cancel exactly. This means that if the incoming electric wave was in the $\boldsymbol{i}$ direction, it'll now be in the $-\boldsymbol{i}$ direction.
- The Poynting vector must also reverse, and since we've already reversed the $\boldsymbol{E}$ field, this means that the $\boldsymbol{B}$ field must *stay the same*

V. THE POYNTING VECTOR

The *Poynting vector* is a tool that we can use to determine where the energy is flowing in a given configuration of electric and magnetic fields. It is given by

$$\boxed{\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}}$$

A thorough justification isn't that difficult, but it requires some pretty nifty vector calculus.

The *Poynting vector* has units of *power per unit area.* Integrating the vector over the area of interested gives the total power flowing through that area.

$$P_{\text{total}} = \iint \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{A}$$

Let's see how the Poynting Vector applies in a few cases

A. Electromagnetic waves

We mentioned above that in electromagnetic waves, the electric and magnetic fields are at right angles. We can therefore calculate the *Poynting vector* to check what direction the energy is flowing (ie: the direction the wave is flowing).

It's important for you to realise that there are three things here

- The direction of the $\boldsymbol{E}$ field.
- The direction of the $\boldsymbol{B}$ field.
- The direction the wave is flowing.

If you know *any* two of those, you can find the third.

Let's take a quick example. Imagine you're told that the $\boldsymbol{E}$ field looks like

$$\boldsymbol{E} = E_0 \sin(kx - \omega t)\boldsymbol{j}$$

Find the $\boldsymbol{B}$ field.

Well, the steps you should be going through in your head are

- The bit before the ωt is a minus, and the letter in the bracket is an x, so the wave is travelling in the *positive x direction.*
- The $\boldsymbol{E}$ field is clearly in the *positive y direction.*

You have two things, therefore you can find the third. It turns out that $\boldsymbol{B}$ is in the *positive z direction.* To find its magnitude, we simply use $\frac{E_0}{B_0} = c$, and get

$$\boldsymbol{B} = \frac{E_0}{c}\sin(kx - \omega t)\boldsymbol{k}$$

B. Circuit elements

You should also be able to apply this reasoning to simple circuit elements. Let's try it with a few examples

- In a simple wire of non-zero resistance carrying current:
  - The $\boldsymbol{E}$ field points *along* the wire, to 'push the electrons along'. (Note: this is *only* true because the wire has resistance).
  - The $\boldsymbol{B}$ field points *around* the wire (because of the current).

  As a result, energy flows *into* the wire. This makes sense; heat energy is flowing *out* of the wire because of the resistance, and it's gotta come from somewhere – in this case, from the electromagnetic energy coming in.

  You might be somewhat surprised that the direction of power flow isn't *along* the wire. The story is in fact somewhat more complicated, but we'll stop here.

- In a coaxial cable, we have the following
  - The $\boldsymbol{B}$ field points around the inner conductor
  - The $\boldsymbol{E}$ field points from the inner to the outer conductor (because the two are at a different potential.

  As a result, energy flows *along* the cable, as you might expect. The details are in your PSet.

- In a charging solenoid
  - The $\boldsymbol{B}$ field points *along* the inside of the solenoid.
  - The $\boldsymbol{E}$ field arises because the magnetic flux in the solenoid is changing, and runs in concentric circles in the middle of the solenoid.

  We find that the Poynting vector flows *into* the solenoid – since it's charging, energy is going *into it.*

VI. DIFFRACTION AND INTERFERENCE

This really is the easiest part of this exam – if you can master it, you're good!

Before we even begin looking at diffraction patterns, let's clear up a common source of confusion. Every formula you'll see describing a diffraction pattern will usually be expressed in terms of $\sin\theta$ (the angle from the centre of the diffracting object to the point of interest on the screen). However, what you're interested in is y, the distance of the point of interest on the screen from the centre of the screen. It turns out that for small θ and y, they are related as follows

$$\boxed{\sin\theta = \frac{y}{D}}$$

Where D is the distance from the diffracting object to the screen.

Now, let's move on to the actual patterns. The two basic cases you need to know about are the following:

- The double slit (slit separation d), in which maxima occur at

  $$d\sin\theta = m\lambda \Rightarrow \boxed{y = \frac{m\lambda D}{d}}$$

  If the slits are infinitely thin, all maxima are of equal intensity.

  For small m, the distance between the slits is constant and given by

  $$\boxed{\Delta y = \frac{\lambda D}{d}}$$

  Note that the more slits you add, the narrower the maxima become. Thus, with two slits, you get a 'spread' around each maximum. With many slits (for example, a CD you get very sharp dots.

- The *single slit* (slit width a, in which **minima** occur at

  $$a\sin\theta = m\lambda \Rightarrow \boxed{y = \frac{m\lambda D}{a}}$$

  The maxima of this distribution are *not* equally bright. The central one is very bright, and the others much less so.

  The *central maximum* of this diffraction pattern has width

  $$\boxed{\Delta y = \frac{2\lambda D}{a}}$$

  Note also that the formula above implies that the central maximum is twice as large as the other maxima. This is important.

It is useful to remember that the size of the pattern on the screen varies *inversely* as the size of the diffraction objects. For example, if you make slits wider apart, the maxima will become closer together. Similarly, if you make a slit wider, the maxima will become closer.

It is also important to realise that these two patterns rarely arise in isolation. What will usually happen is that you will have a situation involving two slits a distance d apart but each of finite width a. It turns out that the result of doing this is *multiplying* the two patterns above. However, because the distance between two slits (d) is usually much greater than the width of the slits (a), the variation due to the two slits is much faster. Thus, we obtain a simple sine wave from the two slits, multiplied by a much *larger* 'single slit' pattern.

These concepts (as well as the concept of interference in this films) are illustrated much more easily with diagrams, and I'll make sure I go through them in class.