The "No-Slip" Condition in a Rolling Body

Many people came up to me after class today as asked me

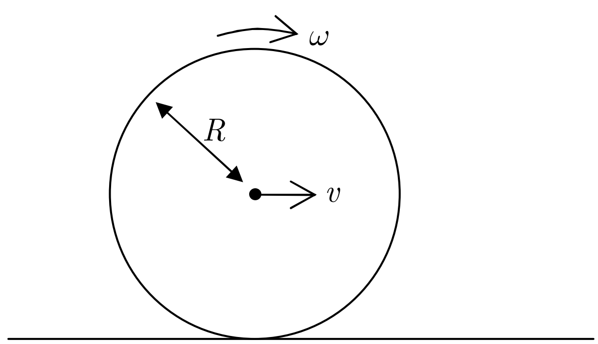
> "*Why is it that when a **body** is **rolling** on a **plane without slipping**, the **point of contact** with the plane **does not move**?*"

The **simple answer** to this question is, quite simply, "*because the **body** does **not slip***"! Why? Because "**slipping**" implies **two bodies** in **contact** moving **relative to each other**. Here, there is **no slipping**. Therefore, the **point of contact** and the **plane** do **not move relative to each other**. Therefore, the point of contact **does not move**.

However, for those still unconvinced, I'll present a **mathematical solution** which should help.

1. The "no-slip" condition

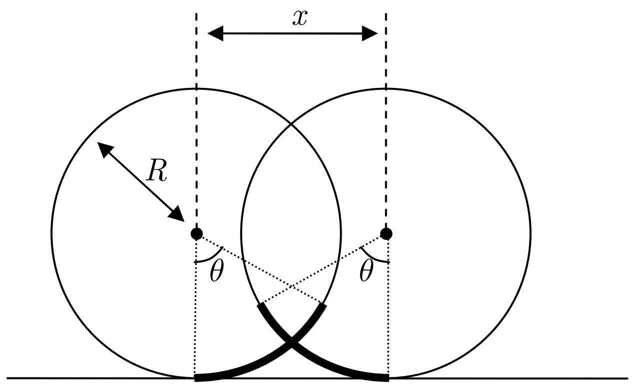
Let's imagine a body **moving forward at speed v on a plane by rolling without slipping**:



As we indicated in the diagram above, the cylinder must also be **rotating** about its **axis**, because it is **rolling**, at an **angular speed** ω.

It is **clear** that ω is **completely determined by v** and **vice-versa**, because if we **change v** (ie: make the cylinder move faster), ω must also **increase** (ie: the cylinder must roll faster). Our **aim** is to find the **relation** between v and ω.

To do this, consider the cylinder **moving forward** a **distance** $\boldsymbol{x}$. As a result, it will **rotate** through an **angle** θ :



Now, consider the following argument

- On purely **mathematical grounds**, the **length** of the **bold arc of the circle** is $R\theta$.
- However, this is **equal** to $\boldsymbol{x}$, because the **distance** the circle has **moved forward** is **equal to** the **arc length**. So $x = R\theta$.

We have thus found

$$\boxed{x = R\theta}$$

Consider **differentiating** each side of this equation (remember R is **constant**)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = R\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

But $\mathrm{d}x/\mathrm{d}t$ is the **velocity**, and $\mathrm{d}\theta\,/\,\mathrm{d}t$ is the **angular velocity**. As such

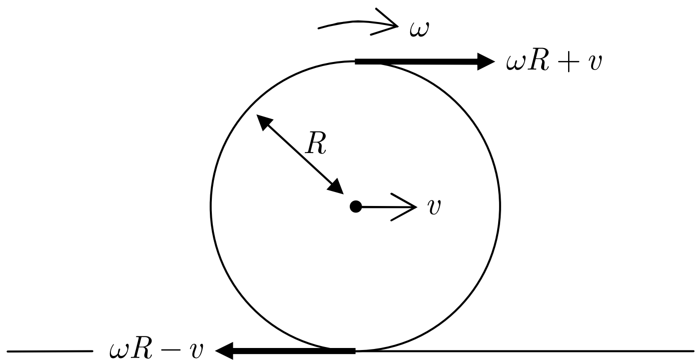
$$\boxed{v = R\omega}$$

These two **boxed items** are the **no-slip conditions**. We have found what we wanted.[1]

[1] Note – this is the same relationship, $v = \omega R$, that we had obtained for **circular motion**. However, it's important to realise that the v in each equation is **different**. In **circular motion**, v refers to a point on the **rim** of the circle. Here, v refers to the **velocity of the centre of mass**. After reading the next section, you should be able to convince yourself that they **have to be the same**, as **predicted** by these equations.

2. The point of contact

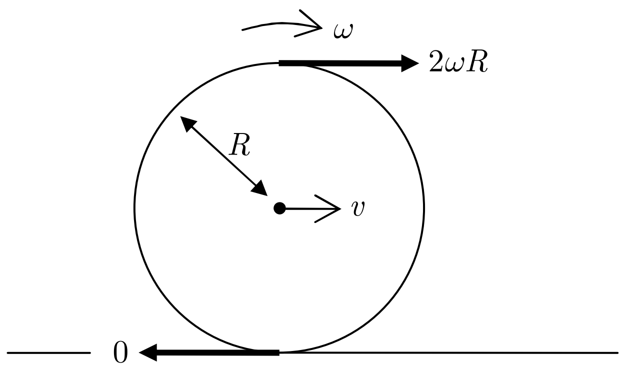
Now, consider a **body** rolling on a **plane** again. We might ask the question "*what are the velocities of the points at the **very top** and **very bottom** of the circle?*" The **answers** are as follows:



Let's see why this is:

- **<u>Top point</u>** – moving **forward** at $\boldsymbol{v}$ (because the **cylinder** is moving **forward** at $\boldsymbol{v}$) but also has an **extra forward speed** ωR because of the **rotation**.
- **<u>Bottom point</u>** – moving **forward** at $\boldsymbol{v}$ (for the same reason) but is now moving **backwards** at ωR because of the **rotation** (think about it – because the circle is rotating *clockwise*, it's moving *back* at its *bottom point*).

However, we found above that $\boxed{v = \omega R}$. Feeding this into our diagram



As indeed expected, we find that the **velocity** of the **point of contact** is **0**. It **does not move**.