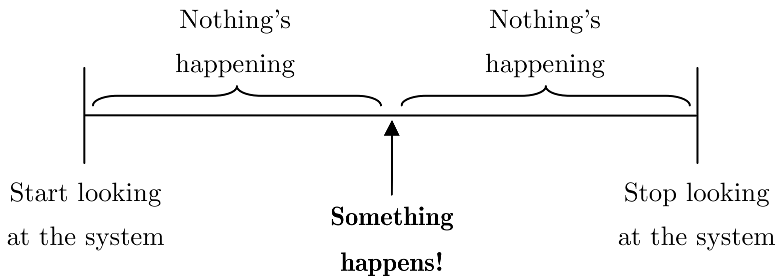
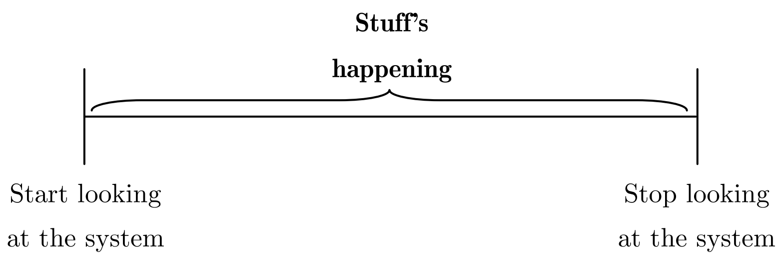
Continuous Mass Transfer

1. Introduction

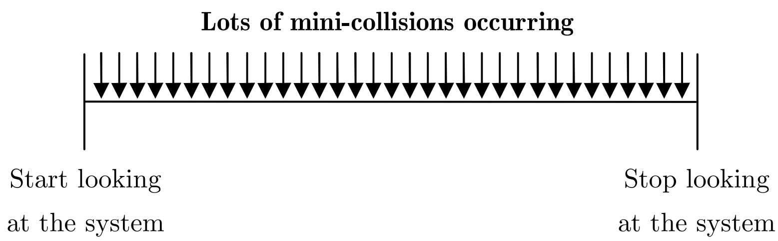
- In dealing with **collisions**, we were dealing with events happening at **particular points in time**. On a timeline, it'd look something like this



- We're now going to deal with a more complicated situation, in which **collisions** are happening **continuously**. Things are happening like this



- This problem is obviously much more difficult, and we want to make it easier. We do that by **splitting the problem** into lots of **mini-collisions**:



  We assume that each collision **occurs** over a **very small time dt**. As we make **dt smaller and smaller**, and the collisions become **closer and closer** to each other, we end up with the **continuous case**.

- This gives us a **differential equation**, which effectively **describes** how the system **changes** with time. The differential equation can then be **solved**.

Daniel Guetta – guetta@mit.edu

- The general method to solve any such problem is therefore
    1. **Find** the **mini-collision** and **set up "before"** and **"after"** diagrams for the collision.
    2. Treat the mini-collision as a normal collision – **apply "External Impulse = Change of momentum"**.
    3. Make **dt smaller** and **smaller** until we get the **continuous case**. Get a differential equation.
    4. **Solve** the **differential equations** obtained.
- The handout will guide you through each of these steps one by one.

2. Setting up the problem – finding the mini-collision

- The first skill we have to get clear is **setting up the problem**, and deciding **what mini-collision** we need to take into account.
- This is the **hardest part**, because it's the part that involves all the physics.
- The **trickiest** part is to correctly determine the **velocities** in the problem. Think **carefully** – the following questions might help:
    - What **frame** are you drawing your diagram **with respect to**? The **ground frame**? The **frame of the object**?
        - In some cases, it makes the **algebra** easier to draw the system **in the frame of the object**.
        - It is usually **conceptually easier**, though, to draw the system **in the ground frame**.

      In **example 2** below, we'll show you both options.
    - Are you given velocities **relative to the ground** (as in **example 1** below) or **relative to something else in the system** (as in **example 2** below)?
    - Are you considering the right **dimension** (ie: do you care about *vertical* or *horizontal* motion? For example, if a **cart** is on a **road** and you're looking to see how **fast** it'll **move forward** as a result of stuff **falling into it**, the **vertical velocity** of the stuff falling into it hardly matters. See **example 3** for a real-life case.
- We'll do a few examples to show you how it's done.

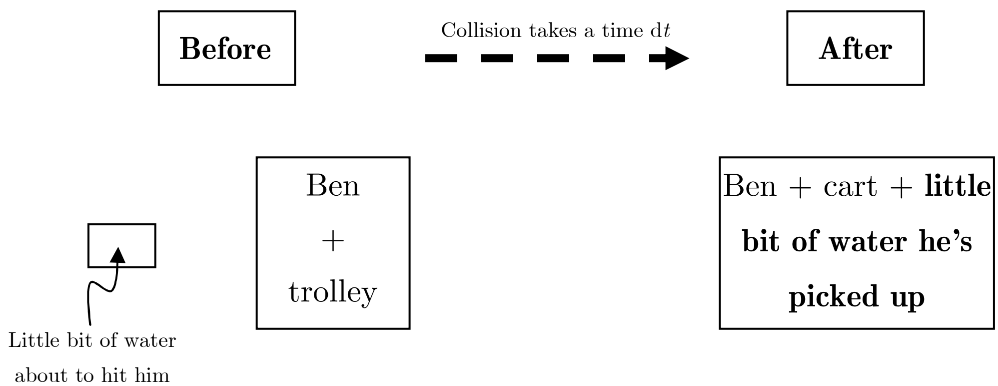
2.1 – Dealing with the simplest kind of problem

<u>Example 1</u>

Ben is standing in a trolley at rest (combined mass is m_b). Katie is standing on the ground with a powerful hose directed towards Ben, and she starts continuously spraying him with water, which hits him at a speed u and falls into the trolley. What mass of water must Katie spray to get Ben moving at a speed V?

What's happening in this case? Ben is **continuously colliding** with an incoming jet of water, picks up the water, and, as a result **speeds up**.

We can **split this** into lots of **mini-collision**, each involving a **tiny bit of water** hitting Ben and taking a **time d*t***. What would the situation look like **before** and **after** this mini-collision? We draw our diagram **with respect to the ground**:



Time to introduce some symbols – what do we know about the situation before the collision, and after?

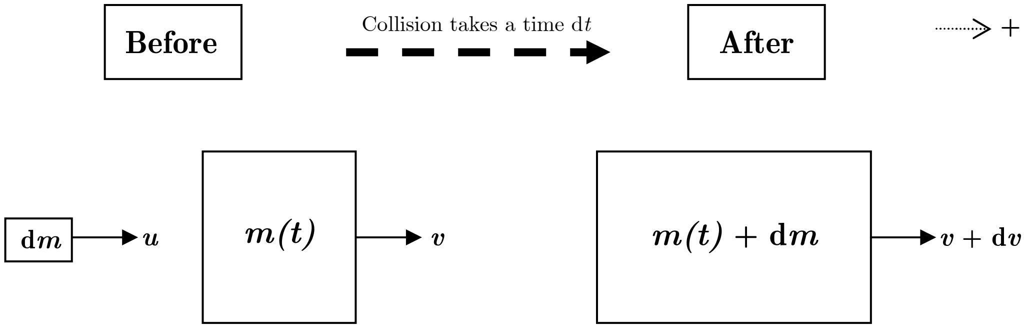
- <u>**Before**</u>
  - We know the **tiny bit of water** is moving at a **speed u** (because we're told in the question) and we denote the small bit of mass **dm**.
  - We **denote** the **mass** of Ben as $m(t)$ and his **speed** as $v(t)$[1]

[1] ***<u>Common mistake</u>*** – a **common mistake** here is to assume that

- Ben's speed **before** is 0 (because the question tells us he starts at rest)

- **After**
  - As a result of the collision, Ben has picked up water, so his mass has increase to $\boldsymbol{m(t) + \mathrm{d}m}$.
  - As a result of the collision, Ben's speed has also increase – and we'll call his new speed $\boldsymbol{v + \mathrm{d}v}$.[2]

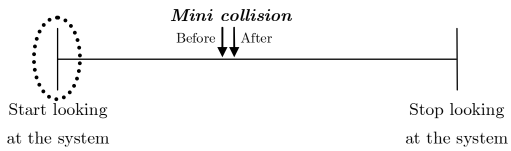
So, our fully-fleshed out diagram for the mini-collision looks like



Which is what we wanted.

- His **mass** before is $\boldsymbol{m_b}$ (before anything has happened, the combined mass of Ben and the trolley is m_b).

However, this is *wrong*. The reason is that these conditions are true at the **start of the problem** – indicated by the dotted circle in this diagram:



However, this is **not** what we want! We want the points "before" and "after" **any** mini-collision at **any** time in the motion (the two black arrows). At those points, Ben **will be moving**, and his **mass** will have **increased** due to water that was sprayed since the start.

[2] Many people are confused by the fact the variable $\boldsymbol{u}$ doesn't appear in the speed after the collision – surely, the speed at which Ben and the trolley move depends on how fast the water hit them. Of course, this is true – but remember that $\mathbf{d}\boldsymbol{v}$ depends on $\boldsymbol{u}$!

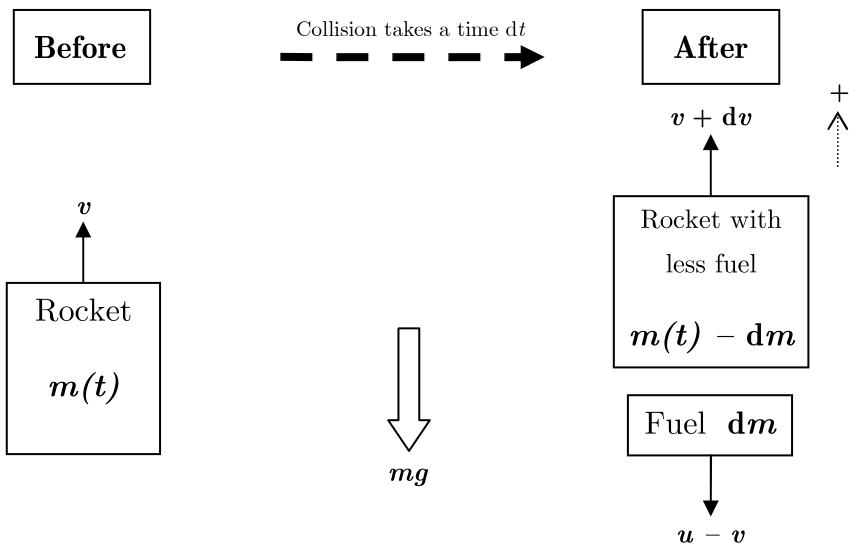
2.2 – Dealing with relative velocities & Impulses

We'll deal with both these situations with one example:

Example 2

A rocket moves upwards in a **constant** gravitational field of strength g by ejecting fuel at a rate σ kg/s downwards, at a speed u relative to the rocket. The rocket has mass M, and originally contains m_0 kg of fuel. The rocket starts at rest. How long till the rocket reaches its escape velocity?

Again, there is a continuous collision between the rocket and the fuel leaving the rocket. We split it into a number of mini-collisions, and ask what things look like before and after each mini-collision. **We first do this with respect to the ground frame**:



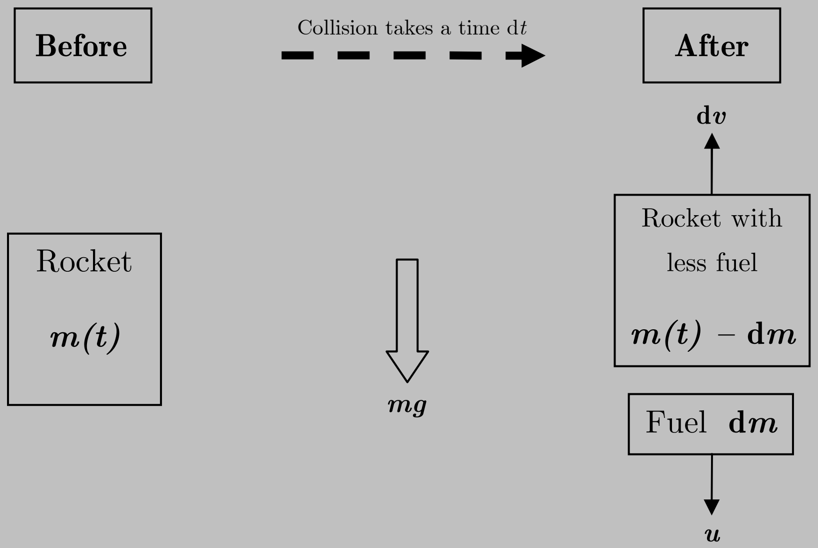
And this is effectively what we wanted. Let's understand why the speed of the ejected fuel is $\boldsymbol{u} - \boldsymbol{v}$:

- The speed with respect to the rocket is $\boldsymbol{u}$ **downwards**, and the rocket is moving **upwards** with speed $\boldsymbol{v}$. So the **resulting** speed **downwards** is $\boldsymbol{u} - \boldsymbol{v}$.
- You should have been **bothered**, though, by the statement that "*the rocket is moving upwards with speed v*" – surely, it's moving upwards with a speed that's **between** $\boldsymbol{v}$ and $\boldsymbol{v} + \mathbf{d}\boldsymbol{v}$. So how can we justify **dropping** the $\mathbf{d}\boldsymbol{v}$? The reason is that the mass we're talking about (the ejected fuel) has mass $\mathbf{d}\boldsymbol{m}$. So if we were to **keep** the $\mathbf{d}\boldsymbol{v}$ in the velocity, all we'd be adding to the **momentum** is a $\mathbf{d}\boldsymbol{m}\mathbf{d}\boldsymbol{v}$

term. This is the **product** of two **infinitesimal quantities**, and which can therefore be **ignored** (as we'll see below).

This is **exactly** the same reason we use $\boldsymbol{m}$ as opposed to $\boldsymbol{m} + \mathbf{d}\boldsymbol{m}$ in our expression for the **force** on the system. Later on, you'll see that to find **impulse**, we have to multiply this **force** by $\mathrm{d}t$, and so the $\mathrm{d}m\ \mathrm{d}t$ terms will vanish.

Note that in this case, it also **makes sense** to draw the system **in the frame of the rocket**. As you'll see later, this can make the **algebra much easier**. In that frame, the system looks like this:



The concept of this frame is pretty **subtle**, so let's flesh it out

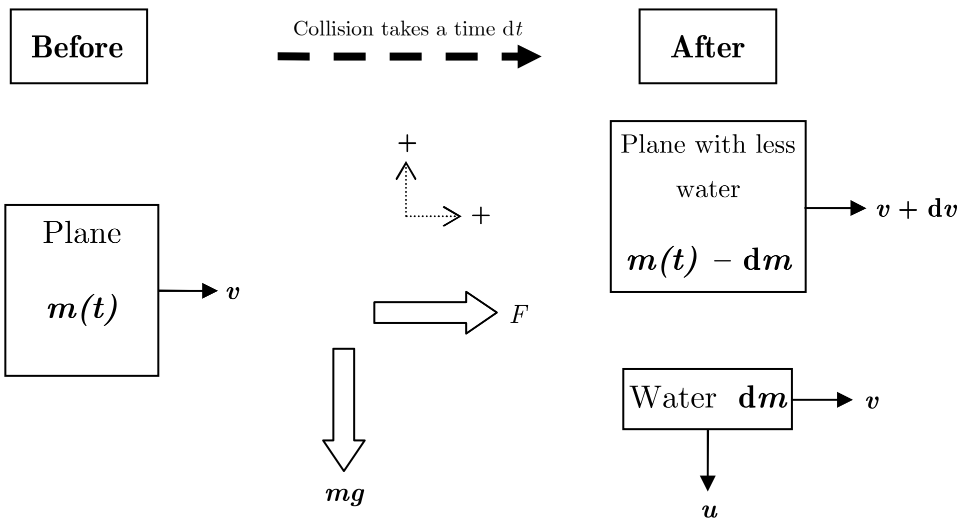
- Remember that we're considering a **mini-collision** here, which could occur at **any point in the motion**. The frame we're considering is one which is **moving with the rocket at the start of <u>that particular mini-collision</u>**. Therefore, the frame we're considering will **change** throughout the motion, because the rocket is **speeding up**. But this is fine, because we can watch each collision in **any frame**.
- In spite of that, however, we must assume that the frame **remains the same** during the **small** interval of time $\mathbf{d}\boldsymbol{t}$. This becomes more and more **accurate** as $\mathrm{d}t$ gets **smaller** and smaller. We do this for **exactly the same reason** we **ignored** $\mathbf{d}\boldsymbol{v}$ when working out the relative velocity above

2.3 – An example with a confusing relative velocity

Example 3

A plane flies over a forest fire and sprays water over it in an attempt to extinguish it. The crew member in the plane sprays the water directly downwards at a speed u and at a rate σ kg/s. The plane's motor provides a constant **horizontal** thrust force F and the plane's original *horizontal* velocity is v_0. Given that the plane has mass M and is originally loaded with a mass of water m_0, find the plane's horizontal velocity a time t later.

This problem is slightly less obvious. The **continuous collision** is between the **plane** and the **ejected fluid**. The diagram **in the ground frame** is

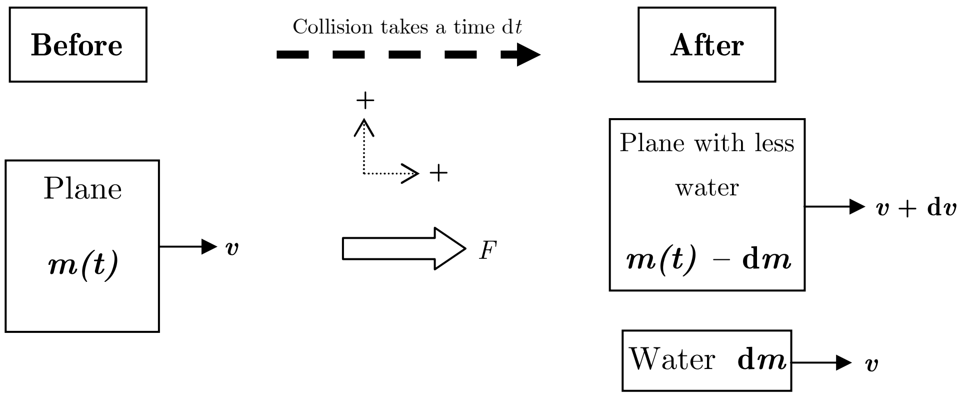


Let's make two important points regarding this diagram

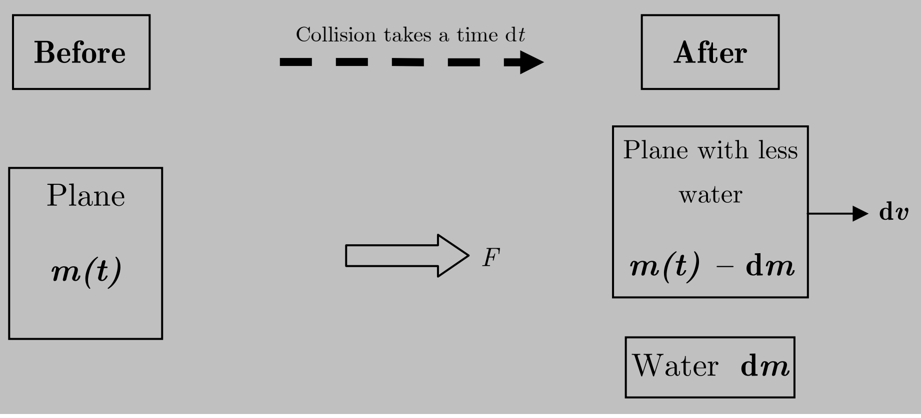
- Notice that even though the question doesn't mention it, the bit of water that leaves the plane has a vertical speed v (because it's *released* from the plane).[3]
- We've included the quantities mg and u in this diagram. However, that was just to check if you were paying attention, because there is <u>*no*</u> reason those should be in there! This problem is only concerned with **horizontal** motion, and so only **horizontal** quantities are relevant.

[3] Once again, we don't write $v + \mathrm{d}v$ for the same reasons as above…

The diagram we want therefore looks like



Once again, it could make the algebra simpler to draw it in the frame of the plane:



2.4 – An example with a confusing question!

This is one of the hardest problems you're likely to encounter

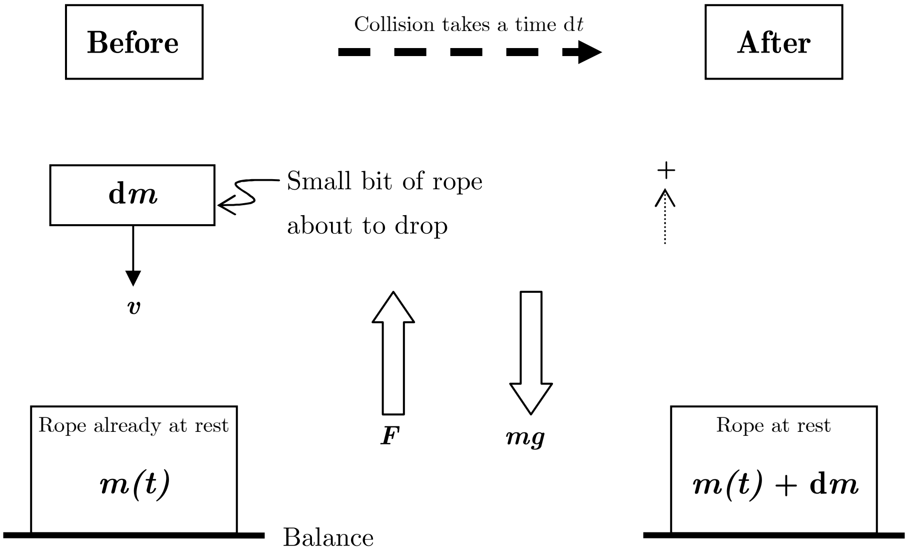
<u>Example 4</u>

A rope of length L and density ρ kg/m is dropped on a balance. What weight will the balance display as a function of time?

The first question you should be asking yourself is "what is this question asking me for"? It wants the weight the balance will display, but what does that mean physically? *<u>Answer</u>*: it wants the **force** that will be acting on the balance pan – because this is what causes the balance to record weight.

This actually makes sense – the **rope** is **constantly colliding** with the **balance** and **being brought to rest** by it, and so we'd expect the **balance** to **exert some impulse** as a result of these collisions, to bring **each small bit of rope to rest**. This will affect the weight reading.

Armed with this knowledge, let's consider a **mini-collision**. This will clearly occur between a **small piece of rope** that's **about to be brought to rest**, and the **rest of the rope** that's already **at rest on the balance**



Again, let's flesh out this diagram with a few comments:

- We have denoted by $\boldsymbol{F}$ the force exerted by the balance on the rope during the collision.
- In this particular case, we can find $\boldsymbol{v}$ in terms of $m(t)$ or t by using simple **kinematics** (the rope is **falling under gravity**). This will be useful later on.
- We have also written down the force of **gravity** acting on the system. For exactly the same reason as above, we have not included **dm** in this expression, because the term will eventually **disappear** when we multiply it by **dt** to get the **impulse**.

3. Applying "Impulse = Change of Momentum"

We're now ready to apply "Impulse = Change of Momentum". This is usually the simplest step and embodies the physics of the problem. We need to consider **both sides** of this equation

- To find the **change in momentum**, simply find the momentum of the system **after the mini-collision** and subtract the momentum **before the mini collision**.
- To find the **impulse**, simply multiply the **force acting** by the **time interval over which it acts** ($\mathrm{d}t$). We assume that $\mathrm{d}t$ is **so small** that the force **does not vary** during that time.
- In all cases, if you ever encounter the **product** of two **infinitesimal quantities** (for example, $\mathrm{d}v\ \mathrm{d}t$, you can just **ignore it**). Superficially, this is because something **very small** multiplied by something **very small** gives something **absolutely tiny** which can be ignored. If you take more advanced **analysis** classes (course 18), you'll see why this is justified.

As usual, the best way to learn is by example, so let's go!

2.1 – Example 1

In this case, there is no impulse on the system. As such, using the diagram

$$\text{Change in momentum} = 0$$
$$\text{Momentum after} - \text{Momentum before} = 0$$
$$\left(m + \mathrm{d}m\right)\left(v + \mathrm{d}v\right) - \left(u\mathrm{d}m + vm\right) = 0$$
$$\cancel{mv} + m\mathrm{d}v + v\mathrm{d}m + \cancel{\mathrm{d}m}\mathrm{d}v - u\mathrm{d}m - \cancel{mv} = 0$$
$$\boxed{m\mathrm{d}v + (v - u)\mathrm{d}m = 0}$$

2.2 – Example 2

This time, we *do* have an impulse acting on the system. The force acting is $-mg$, and so the impulse will be $-mg\ \mathrm{d}t$. As such

$$\text{Change in momentum} = -mg\mathrm{d}t$$
$$\text{Momentum after} - \text{Momentum before} = -mg\mathrm{d}t$$
$$\left(v + \mathrm{d}v\right)\left(m - \mathrm{d}m\right) - \left(u - v\right)\mathrm{d}m - \left(mv\right) = -mg\mathrm{d}t$$
$$\cancel{mv} - \cancel{v\mathrm{d}}m + m\mathrm{d}v - \cancel{\mathrm{d}m}\mathrm{d}v - u\mathrm{d}m + \cancel{v\mathrm{d}}m - \cancel{mv} = -mg\mathrm{d}t$$
$$\boxed{m\mathrm{d}v - u\mathrm{d}m = -mg\mathrm{d}t}$$

Note that if we had directly used the **second diagram** in which we considered the situation **in the frame of the rocket**, our equations would have looked like

$$\text{Change in momentum} = -mg\mathrm{d}t$$
$$\text{Momentum after} - \text{Momentum before} = -mg\mathrm{d}t$$
$$\left(m - \mathrm{d}m\right)\mathrm{d}v - u\mathrm{d}m - \left(0\right) = -mg\mathrm{d}t$$
$$m\mathrm{d}v \cancel{- \mathrm{d}m\mathrm{d}v} - u\mathrm{d}m = -mg\mathrm{d}t$$
$$\boxed{m\mathrm{d}v - u\mathrm{d}m = -mg\mathrm{d}t}$$

This is **exactly** the same result as above. However, I'm sure you'll agree it was much easier to obtain algebraically… This is what often makes the second method useful.

2.3 – Example 3

Again, there is an impulse on the system. We're interested in the **horizontal** dimension, and the impulse is $F\,\mathrm{d}t$. As such, we get

$$\text{Change in momentum} = F\mathrm{d}t$$
$$\text{Momentum after} - \text{Momentum before} = F\mathrm{d}t$$
$$\left(m - \mathrm{d}m\right)\left(v + \mathrm{d}v\right) + v\mathrm{d}m - mv = F\mathrm{d}t$$
$$\cancel{mv} + m\mathrm{d}v - \cancel{v\mathrm{d}m} - \cancel{\mathrm{d}m\mathrm{d}v} + \cancel{v\mathrm{d}m} - \cancel{mv} = F\mathrm{d}t$$
$$\cancel{mv} + m\mathrm{d}v - \cancel{v\mathrm{d}m} - \cancel{\mathrm{d}m\mathrm{d}v} + \cancel{v\mathrm{d}m} - \cancel{mv} = F\mathrm{d}t$$
$$\boxed{m\mathrm{d}v = F\mathrm{d}t}$$

Once again, if we'd used the **second diagram** of the situation **in the frame of the plane**, we'd have obtained

$$\text{Change in momentum} = F\mathrm{d}t$$
$$\text{Momentum after} - \text{Momentum before} = F\mathrm{d}t$$
$$\left(m - \mathrm{d}m\right)\mathrm{d}v - 0 = F\mathrm{d}t$$
$$m\mathrm{d}v - \cancel{\mathrm{d}m\mathrm{d}v} = F\mathrm{d}t$$
$$\boxed{m\mathrm{d}v = F\mathrm{d}t}$$

Once again, exactly as above.

2.4 – Example 4

This time, our impulse is due to *two* forces, and so

$$\text{Change in momentum} = \left(F - mg\right)\mathrm{d}t$$
$$\text{Momentum after} - \text{Momentum before} = \left(F - mg\right)\mathrm{d}t$$
$$0 - \left(v\mathrm{d}m\right) = \left(F - mg\right)\mathrm{d}t$$
$$\boxed{v\mathrm{d}m = \left(F - mg\right)\mathrm{d}t}$$

4. Moving to the continuous case by making $\mathrm{d}t \to 0$

Moving to the continuous case is a **no-brainer**. All you need to do is **divide the equation obtained by d*t***. As $\mathrm{d}t$ tends to 0, the ratios of infinitesimal quantities tend to **differentials**. If you were doing a **more advanced math course**, we'd have to **think more carefully** about what you were doing (using **limits, etc…**). But for 8.01, just **divide** by **d*t*** and you'll get your differential equation!

However, there is a **<u>crucial point</u>** that is **easy to miss** – and that is to **keep track of what the** "m" in your equations above refers to. Read this carefully:

- **Usually**, in your **final equation**, you want ***m*** to refer to the mass of the **rocket/cart/plane/spaceship**, etc…, because this is the **mass** of **interest** in the question.
- **However**, in your equations above, **d*m*** does **<u>not</u>** refer to that – it refers to the **small mass being ejected/taken in**.
- Now, it's clear that this **won't make a difference** to the **magnitude** of **d*m***, because the **change in mass of the rocket** is **equal to** whatever is **ejected/taken in**.
- However, it **might** make a difference to the **sign** of **d*m***. Consider [I'll assume we're using a rocket, but replace "rocket" by "cart/plane/etc…" where appropriate!]
    - If mass is **taken in**, then $\mathrm{d}m_{\text{taken in}} = \mathrm{d}m_{\text{rocket}}$ because an **<u>increase</u> in amount of material taken in** is equivalent to an **<u>increase</u> in the rocket mass**.
    - However, if mass is **ejected**, then $\mathrm{d}m_{\text{ejected}} = -\mathrm{d}m_{\text{rocket}}$ because an **<u>increase</u> in amount of material ejected** is equivalent to a **<u>decrease</u> in rocket mass**.
- We'll have to be extremely careful, in the calculations below, to use the right sign.

3.1 – Example 1

$$m\mathrm{d}v + (v-u)\mathrm{d}m = 0$$

$$\boxed{m\frac{\mathrm{d}v}{\mathrm{d}t} + (v-u)\frac{\mathrm{d}m}{\mathrm{d}t} = 0}$$

This is correct, because water is being **added** to the cart.

3.2 – Example 2

$$m\mathrm{d}v - u\mathrm{d}m = -mg\mathrm{d}t$$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} - u\frac{\mathrm{d}m}{\mathrm{d}t} = -mg$$

However, we need to **flip the sign**, because stuff is **leaving** the rocket

$$\boxed{m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t} = -mg}$$

3.3 – Example 3

$$m\mathrm{d}v = F\mathrm{d}t$$

$$\boxed{m\frac{\mathrm{d}v}{\mathrm{d}t} = F}$$

[Note that we've just **recovered $\boldsymbol{F = ma}$**! The mass dropping out of the plane has no effect. Can you think why? Think of the Calvin and Hobbs problem in Test 1]. There's no sign splitting to do, because $\mathrm{d}m$ doesn't appear here!

3.4 – Example 4

$$v\mathrm{d}m = \left(F - mg\right)\mathrm{d}t$$

$$\boxed{v\frac{\mathrm{d}m}{\mathrm{d}t} = F - mg}$$

In this case every extra bit of chain is **adding** to the lump of chain, so the sign is right.

3.5 – Boring note

It might be useful for you to sit and reflect on the relevance of what we've obtained. We found a **differential equation**. What that tells us is **how things change** over a **small time interval**. In other words, it's a **very concise mathematical statement** of **the effect**

of a mini-collision. We now need to find a way to **sum all these effects** over the **whole motion**… enter **integration** ☺!!

5. Solving the differential equation

Solving the differential equation involves **no physics** and some **horrible maths**. We'll try and make the process as **painless as possible**!

To solve these problems, we note that there are always **three variables** involved

- The **mass** of the thing
- The **velocity** of the thing
- The **time** that has elapsed.

Most problems you'll be given will require you to do one of three things

1. Find a **relation** between **mass** and **velocity** (For example "*find the speed of the rocket after it's ejected all its fuel*" or "*find the velocity of the rocket as a function of its mass*", etc…) – **example 1**.
2. Find a **relation** between **velocity** and **time** (For example: "*find the speed of the cart 10 seconds after it has been launched*" or "*how long will the rocket take to reach the earth's escape velocity*") – **examples 2** and **3**.
3. Find something to do with the **force/impulse** – **example 4**.

Cases 1 and 2 are very similar and involve the **same** first steps, but case 2 involves an **extra step** at the end… Case 3 is completely different. Let's cover **each in turn**.

5.1 – Finding the relationship between velocity and mass

Finding the relationship between velocity and mass involves **two simple steps**

1. **Separate variables**
2. **Integrate** – this **must be *with respect to time***, because when **adding up** all our **mini-collisions**, we're summing over a **continuum of time**.
3. **Pick limits**

Again, it makes sense to show you how to do this **by example**… The first example will be the **most detailed** to describe the **principle**, and then we'll go **faster**.

5.1.1 – Example 1

The differential equation we ended up with was

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = (u-v)\frac{\mathrm{d}m}{\mathrm{d}t}$$

Now, consider – what would happen if we just integrated with respect to time?

$$\int m\frac{\mathrm{d}v}{\cancel{\mathrm{d}t}}\cancel{\mathrm{d}t} = \int (u-v)\frac{\mathrm{d}m}{\cancel{\mathrm{d}t}}\cancel{\mathrm{d}t}$$
$$\int m(t)\,\mathrm{d}v = \int (u-v(t))\,\mathrm{d}m$$

We have a problem – this integral **cannot be done**. Why? Because m is **not a constant** – it's a **function of time** – **and so is v**. Therefore, when we **integrate** with respect to **another variable**, the integral is **not easy**.

To **fix this problem**, we need to go back to the differential equation and **re-arrange it** so that **each side is a function of one variable only**. In this case, our differential equation was

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = (u-v)\frac{\mathrm{d}m}{\mathrm{d}t}$$

Let's **divide** both sides by $\boldsymbol{m}$ and $\boldsymbol{u} - \boldsymbol{v}$. We then get

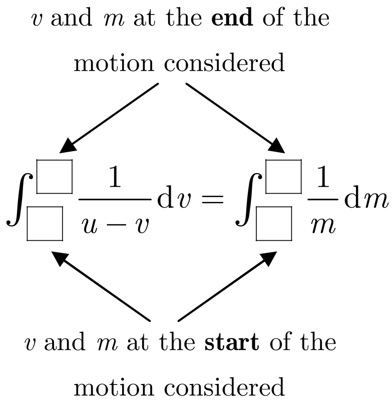
$$\frac{1}{u-v}\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}t}$$

Now, integrate with respect to time

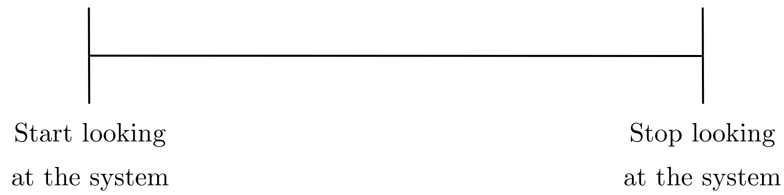
$$\int \frac{1}{u-v}\,\mathrm{d}v = \int \frac{1}{m}\,\mathrm{d}m$$

And now this integral **<u>can</u>** be done, because u is a constant and we can easily integrate m and v with respect to **themselves**.

Now, let's think of the **limits** of integral the general rule with limits is as follows:



What do we mean by "**start**" and "**end**" of the motion considered? We mean the two **end-points** of the motion:



We are **no longer** considering **mini-collisions**. We are considering **the whole event**.

So, in this particular problem, we are told

- That the cart starts at rest
- That the original mass of the cart+ben is m_b.

We want to know **how much water** needs to be sprayed for it to reach a **final speed** $\boldsymbol{V}$. In other words:

- The **final speed** we want in the equation above is $\boldsymbol{V}$.
- The **final mass** we want is $\boldsymbol{m_b + M}$, where M is the unknown amount of water.

So:

$$\int_0^V \frac{1}{u-v}\,\mathrm{d}v = \int_{m_b}^{m_b+M} \frac{1}{m}\,\mathrm{d}m$$
$$\Big[-\ln\big(u-v\big)\Big]_0^V = \Big[\ln m\Big]_{m_b}^{m_b+M}$$

[Note the minus sign in the LHS integral!]

$$-\ln\left(u-V\right)+\ln\left(u\right)=\ln\left(m_b+M\right)-\ln m_b$$
$$\ln\left(\frac{u}{u-V}\right)=\ln\left(\frac{m_b+M}{m_b}\right)$$
$$\frac{u}{u-V}=\frac{m_b+M}{m_b}$$
$$\boxed{M=m_b\left(\frac{u}{u-V}-1\right)}$$

And this is our answer.

5.1.2 – Example 2

The differential equation we had obtained was

$$m\frac{\mathrm{d}v}{\mathrm{d}t}+u\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

To **separate variables**, all we really need to do is **divide by m**

$$\boxed{\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{u}{m}\frac{\mathrm{d}m}{\mathrm{d}t}=-g}$$

At this point, we'll **pause**, because the problem asks us to find **velocity** in terms of **time**, and so a **slightly different method is needed**, which we'll cover in the **next section**.

5.1.3 – Example 3

The differential equation we had was

$$m\frac{\mathrm{d}v}{\mathrm{d}t}=F$$

Again, dividing by m does the trick:

$$\boxed{\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{F}{m}}$$

5.2 – Finding the relationship between velocity and time

For these problems, an **extra bit of information** is needed – namely, we need to know **how m changes with t**. Using this information, we can **find $\mathrm{d}m/\mathrm{d}t$**, and feed it into our equation **before we integrate**.

5.2.1 – Example 2

We found, for this example, that the **differential equation** was

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{u}{m}\frac{\mathrm{d}m}{\mathrm{d}t} = -g$$

We are told in the question that "**the rocket ejects fuel at a rate σ kg/s**". This means that

$$m = \overbrace{M + m_0}^{\text{Original mass of fuel}} - \overbrace{\sigma t}^{\text{Amount ejected}}$$

$$\boxed{\frac{\mathrm{d}m}{\mathrm{d}t} = -\sigma}$$

Putting this into the equation above

$$\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{u}{m}\sigma = -g$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{u}{M + m_0 - \sigma t}\sigma = -g$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{u\sigma}{M + m_0 - \sigma t} - g$$

Now, we can integrate with respect to t

$$\int \frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t = \int \frac{u\sigma}{M + m_0 - \sigma t} - g\,\mathrm{d}t$$

$$\int \mathrm{d}v = \int \frac{u\sigma}{M + m_0 - \sigma t} - g\,\mathrm{d}t$$

To find the **limits**, we note that

- The rocket **starts** at **rest** and needs to reach a **final speed** $\boldsymbol{v_{esc}}$
- The rocket **starts** at $\boldsymbol{t = 0}$ and **ends** at a time $\boldsymbol{T}$, which we need to find

So

$$\int_0^{v_{esc}} \mathrm{d}v = \int_0^T \frac{u\sigma}{M + m_0 - \sigma t} - g\,\mathrm{d}t$$

$$\left[v\right]_0^{v_{esc}} = \left[-u\ln\left(M + m_0 - \sigma t\right) - gt\right]_0^T$$

$$v_{esc} = -u\ln\left(M + m_0 - \sigma T\right) - gT + u\ln\left(M + m_0\right)$$

$$\boxed{v_{esc} = u\ln\left(\frac{M + m_0}{M + m_0 - \sigma T}\right) - gT}$$

Which we *could* find T from numerically.

5.2.2 – Example 3

For example 3, the differential equation we obtained was

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{F}{m}$$

In this case

$$m = M + m_0 - \sigma t$$

And so

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{F}{M + m_0 - \sigma t}$$

We know that

- The **original velocity** of the plane is $\boldsymbol{v_0}$ and it's **final velocity** is an **unknown** $\boldsymbol{V}$.
- The plane **starts** at $\boldsymbol{t = 0}$ and ends at $\boldsymbol{t = T}$ (a variable in the problem).

So

$$\int_{v_0}^{V} \mathrm{d}v = \int_0^T \frac{F}{M + m_0 - \sigma t}\,\mathrm{d}t$$

$$V - v_0 = \left[-\frac{F}{\sigma}\ln\left(M + m_0 - \sigma t\right)\right]_0^T$$

$$V - v_0 = -\frac{F}{\sigma}\ln\left(M + m_0 - \sigma T\right) + \frac{F}{\sigma}\ln\left(M + m_0\right)$$

$$\boxed{V = v_0 + \frac{F}{\sigma}\ln\left(\frac{M + m_0}{M + m_0 - \sigma T}\right)}$$

Which is precisely what we wanted.

5.3 – Finding something about the force/impulse

In example 4, we found that

$$v\frac{\mathrm{d}m}{\mathrm{d}t} = F - mg$$

$$F = v\frac{\mathrm{d}m}{\mathrm{d}t} + mg$$

We noted above, however, that what we want to find in this case is $\boldsymbol{F}$ – because this will indicate the **weight** that the balance will read. To do this, then, we must find expressions for the quantities on the RHS… Let's do it!

- ***Find an expression for v in terms of t***

  To do this, we must first ask – ***what*** is $\boldsymbol{v}$? Answer: $\boldsymbol{v}$ is the **velocity** of the piece of the chain that's **about to hit** at a **time** $\boldsymbol{t}$.

  So – let's look at that "small piece of chain" and **follow its trajectory**

  - It starts off **at rest before the chain is dropped**.
  - It is accelerated by **gravity only**, and so we can use **kinematics** to find its **velocity** when it reaches the balance after a time t

  $$v_f = v_0 + at$$
  $$v_f = gt$$

  And so

  $$\boxed{v(t) = gt}$$

- ***Finding an expression for m in terms of t***

  To do this, we must first ask – ***what*** is $\boldsymbol{m}$? Answer: $\boldsymbol{m}$ is the **amount of rope** that's **already on the balance** at **time** $\boldsymbol{t}$.

  To find it, consider $\boldsymbol{h}$ – the **length** of rope that has **already hit** at **time** $\boldsymbol{t}$. Consider a **small piece of rope about to hit** and consider its trajectory:

  - It started off **at rest before the chain was dropped**, at a **height** $\boldsymbol{h}$.
  - It is accelerated by **gravity only**, and so we can use **kinematics** to find an expression for h

  $$x = v_0 t + \tfrac{1}{2}at^2$$
  $$h = \tfrac{1}{2}gt^2$$

  The **mass** of this amount of rope is ρh, and so

  $$\boxed{m(t) = \tfrac{1}{2}\rho g t^2}$$
  $$\frac{\mathrm{d}m}{\mathrm{d}t} = \rho g t$$

Feeding these items into our equation

$$F = v\frac{\mathrm{d}m}{\mathrm{d}t} + mg$$
$$F = \left(gt \times \rho gt\right) + \left(\tfrac{1}{2}\rho gt^2 \times g\right)$$
$$F = \rho g^2 t^2 + \tfrac{1}{2}\rho g^2 t^2$$
$$F = \frac{3}{2}\rho g^2 t^2$$

And so the mass displayed by the balance is

$$\boxed{m_{\text{displayed}} = \frac{F}{g} = \frac{3}{2}\rho g t^2}$$

Which is precisely what we wanted!