# Game Theory

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Game Theory deals with situations in which the utility of one player depends on the actions on some of the other players as well as their own.

## 1 Basic elements of noncooperative games

**Example**

**Sealed-bid auctions** : First-price auction: the winner pays his valuation. Second-price auction: the winner pays the second highest bid. All pay auction: everyone pays their bid, regardless of whether they win or lose.

**Oligopolist models** : $n > 1$ firms. Each has the ability to produce any quantity $q_i$ of a product at cost $C_i(q_i)$.

**Cournot** : quantity competition. Firms simultaneously and secretly choose their outputs, and the price is determined by the inverse market demand

$$p = p\left(\sigma_i q_i\right)$$

The profit of first $i$ is

$$\Pi_i(q_1, \cdots, q_n) = q_1 p\left(\sigma_i q_i\right) - C_i(q_i)$$

**Bertrandt** : price competition. Firms simultaneously and secretly choose their prices $p_i$. These are then revealed and demand $D(p_m in)$ is realised. All firms who posted that price share in the demand (and profits) equally.

### 1.1 Extensive Representation

The *extensive form* of a game involves a *game tree* – this shows every possible move every player could make, sequentially.

It is possible that, before his move, the player could be at any one of a number of nodes (for example, if the other player hid their action). These nodes are aggregated in an *information set*, a subset of a given player's decision nodes at a given time. Note that every decision node in an information set must have the same set of possible actions.

**Definition 1. (Perfect information)** A game is one of *perfect information* if each information set contains a single decision node. Otherwise, it is a game of *imperfect information*.

Chance can be introduced by adding 'nature' as a player, with a probability associated to each branch.

### 1.2 Normal Form Representation

**Definition 2. (Strategy)** Let

- $\mathcal{H}_i$ denote the collection of player $i$'s information sets.
- $\mathcal{A}$ denote the set of possible actions in the game.
- $C(H) \subset \mathcal{A}$ denote the set of actions possible at information set $H$.

A *strategy* for player $i$ is a function $s_i : \mathcal{H}_i \to \mathcal{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.

Clearly, if we have *every* possible strategy as well as the outcomes from each strategy, we have a full description of the game.

**Definition 3. (Normal form representation)** For a game with $\ell$ players, the normal form representation $\Gamma_N$ specifies (for each player) a set of strategies $\mathcal{S}_i$ and a payoff function $u_i(s_1, \cdots, s_\ell)$ (with $s_i \in \mathcal{S}_i$ for all $i$), giving von Neumann-Morgenstern utility levels associated with the (possibly random) outcome arising from strategies

$s_1, \cdots, s_\ell$. Formally,

$$\Gamma_N = [\ell, \{\mathcal{S}_i\}, \{u_i(\cdot)\}]$$

Every extensive representation has a unique normal form representation. The reverse is not true – some information is lost in the normal form.

### 1.3 Randomized choices

**Definition 4. (Mixed strategy)** Given a player $i$'s (finite) pure strategy set $\mathcal{S}_i$, a *mixed strategy* for this player, denoted $\sigma_i : \mathcal{S}_i \to [0, 1]$, assigns to each pure strategy $s_i \in \mathcal{S}_i$ a probability $\sigma_i(s_i) \geq 0$ that it will be played (with probabilities summing to 1).

(All possible mixed strategies for player $i$ lie on a simplex, which we denote $\Delta(\mathcal{S}_i)$). We therefore denote a normal form game with mixed strategies by

$$\Gamma_N = [\ell, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$$

This randomization over strategies induces a randomization over terminal nodes. Letting $\mathcal{S} = \mathcal{S}_1 \times \cdot \times \mathcal{S}_\ell$, player $i$'s von Neumann-Morgenstern utility from mixed strategy profile $\sigma$ is

$$\sum_{s \in \mathcal{S}} \left[ u_i(s) \prod_{i=1}^{\ell} \sigma_i(s_i) \right]$$

Note that an alternative representation of randomized choices that uses the extensive version of the game is through a *behaviour strategy*, which, for every information set, associates a probability to each action. For games of perfect recall (in which a player remembers what he did in every previous move), Kunh (1953) showed that these two concepts are identical.

## 2 Simultaneous Move Games

Given a a normal form game $\Gamma_N$, can be predict what will happen?

We denote

$$s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_\ell)$$

$$\mathcal{S}_{-i} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \cdots \times \mathcal{S}_\ell$$

### 2.1 Dominant and Dominated Strategies

**Definition 5. (Strictly dominant strategy)** A strategy $s_i \in \mathcal{S}_i$ is a *strictly dominant strategy* for player $i$ if for all $s'_i \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

for all $s_{-i} \in \mathcal{S}_{-i}$.

The strategy $s_i$ is said to be *weakly dominant* if the inequality above is strict for at least *one* $s'_i$, but possibly soft for the others.

Rational players never play strictly dominated strategies. The argument for weakly dominated strategies is not as strong, since there are always some strategies that the opponent might play against which the weakly dominated strategy performs just as well. A weakly dominated strategy can only be declared irrational in a situation in which the player knows his opponent puts positive probability on every strategy available to him.

Note that if we assume full information and, further, that every player knows the other player is rational, we can go through multiple iterations of deleting strictly dominated strategies. The order in which such strategies are deleted does not matter. Deleting *weakly* dominated strategies results in a game that *does* depend on the order of deletion (because the assumption an opponent puts positive probability on all strategies is incorrect if some of those are weakly dominated and will be deleted).

**Definition 6. (Strictly dominant mixed strategy)** A strategy $\sigma_i \in \Delta(\mathcal{S}_i)$ is *strictly dominated* for player $i$ if there exists another strategy $\sigma'_i \in \Delta(\mathcal{S}_i)$ such that for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(\mathcal{S}_j)$

$$u_i(\sigma'_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$$

The following theorem states that to check strict dominance of $\sigma_i$, it is enough to check it against the *pure* strategies of $i$'s opponents

> **Theorem 1.** A strategy $\sigma_i \in \Delta(\mathcal{S}_i)$ is *strictly dominated* for player $i$ if and only if there exists another strategy $\sigma_i' \in \Delta(\mathcal{S}_i)$ such that for all $s_{-i} \in \mathcal{S}_{-i}$
>
> $$u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i})$$

*Proof.* The $\Rightarrow$ part is obvious. For the $\Leftarrow$ part, note that the payoff of any mixed strategy is a convex combination of the payoffs of the pure strategies. □

Note:

- A pure strategy that is not strictly dominated by pure strategies may be strictly dominated by a mixed strategy.
- A mixed strategy may be strictly dominated even if none of the pure strategies it puts weight on are strictly dominated.
- Any strictly dominant mixed strategy must be pure.
- Any mixed strategy that puts weight on a strictly dominated pure strategy is strictly dominated.

**Example**

In a two-person game in which player 2 only has two options, dominance for player 1 can be examined graphically. Let the $x$ and $y$ axis denote the payoff from player 1 for of player 2's strategies. Each one of player 1's strategies can then be plotted on this graph. Mixed strategies can be represented by lines between pure strategies.

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## 2.2 Rationalizable Strategies

**Definition 7. (Best response)** $\sigma_i$ is the best response for player $i$ to his rival's strategies $\sigma_{-i}$ if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(\mathcal{S}_i)$.

The strategy is *never a best response* if there is no $\sigma_{-i}$ for which $\sigma_i$ is a best response.

The strategies in $\Delta(\mathcal{S}_i)$ that survive the iterated deletion of strategies that are never a best response are known as player $i$'s *rationalizable strategies*.

For two-player games, a strategy is strictly dominated if and only if it is never a best response. For larger games, strictly dominated strategies $\subseteq$ never best responses,[1] so by iteratively eliminating non-rationalizable strategies, we end up with a set of strategies no larger than the set resulting from the iterated deletion of strictly dominated strategies. Once again, the order in which non-rationalizable strategies are deleted does not matter.

Under weak condition, a player always has at least one rationalizable strategy.

## 2.3 Nash Equilibrium

**Definition 8. (Nash equilibrium)** A strategy profile $s = (s_1, \cdots, s_\ell)$ constitutes a *Nash equilibrium* if for every $i$

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

for all $s_i' \in \mathcal{S}_i$.

A mixed strategy profile $\sigma = (\sigma_1, \cdots, \sigma_\ell)$ constitutes a *Nash equilibrium* if, for every $i = 1, \cdots, \ell$

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(\mathcal{S}_i)$.

In this equilibrium, each strategy is a best response (in the sense above) to the strategy *actually played* by rivals. A rationalizable strategy is one that is a best response to *any rational* strategy played by rivals. A Nash equilibrium only requires this of the *actual strategy*.

Note that every Nash equilibrium is also rationalizable,

[1] In particular, a strategy may be rendered non-rationalizable by one $\sigma_i'$ in the case of one $\sigma_{-i}$, and another $\sigma_i'$ in the case of another. Neither strategy would strictly dominate it.

so this concept further sharpens our list of acceptable strategies.

Another way to define a Nash equilibrium is using the concept of a player's *best-response correspondence*

**Definition 9. (Best-response correspondence)** A player's *best response correspondence*, $b_i : \Delta(\mathcal{S}_{-i}) \to \Delta(\mathcal{S}_i)$, is defined as

$$b_i(\sigma_{-i}) = \text{argmax}_{\sigma_i \in \Delta(\mathcal{S}_i)} \{u_i(\sigma_i, \sigma_{-i})\}$$

We further let

$$b(\sigma) = \{b_i(\sigma_{-i})\}_{i=1}^{\ell}$$

$\sigma$ is then a Nash equilibrium if $b(\sigma) = \sigma$.

It makes intuitive sense that out of all the pure strategies player $i$ could use against $\sigma_{-i}$, it would only want to use the best ones. This, it turns out, provides an easy way to check whether $\sigma$ is a Nash equilibrium

**Theorem 2.** Let $\mathcal{S}_i^+ \subset \mathcal{S}_i$ denote the set of pure strategies that player $i$ plays with positive probability in mixed strategy profile $\sigma = (\sigma_1, \cdots, \sigma_\ell)$. This profile is a Nash equilibrium if and only if for all $i = 1, \cdots, \ell$

1. $u_i(s_i, \sigma_{-1}) = u_i(s_i', \sigma_{-i})$ for all $s_i, s_i' \in \mathcal{S}_i^+$.
2. $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ for all $s_i \in \mathcal{S}_i^+$ and all $s_i' \notin \mathcal{S}_i^+$.

*Proof.* $\boxed{\Rightarrow}$ If either condition does not hold, then there are strategies $s_i \in \mathcal{S}_i^+$ and $s_i' \in \mathcal{S}_i$ such that $u_i(s_i', \sigma_{-i}) > u_i(s_i, \sigma_{-1})$

$\boxed{\Leftarrow}$ Assume $\sigma$ is not a Nash equilibrium. Then there is some player $i$ who has a strategy $\sigma_i'$ with $u_i(\sigma_i', \sigma_{-i}) > u(\sigma_i, \sigma_{-i})$. This means there must be an $s_i'$ (played with positive probability in $\sigma_i'$) for which $u(s_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$. But since $u_i(\sigma_i, \sigma_{-i} = u_i(s_i, \sigma_{-i})$ for all $s_i \in \mathcal{S}_i^+$, (1) and (2) cannot both be true. □

Each player is indifferent to the weights they put on those best strategies. These are only predicated by the requirement that the *other* pla

**Theorem 3.** Every game $\Gamma_N = [\ell, \{\Delta(\mathcal{S})\}, \{u(\cdot)\}]$ in which the sets $\mathcal{S}_1, \cdots, \mathcal{S}_\ell$ have a finite number of elements has a mixed stategy Nash equilibrium

**Example**

Consider the following game

| | L | R |
|---|---|---|
| U | (3,2) | (0,0) |
| D | (0,0) | (2,3) |

Suppose player $i$ plays their first choice with probability $p_i$.

Player 1's utilities are then

$$u_1(U) = 3p_2 \qquad u_1(D) = 2(1 - p_2)$$

These two options have equal utility when $p_2 = \frac{2}{5}$.

Now consider player 2

$$u_2(L) = 2p_1 \qquad u_2(R) = 3(1 - p_1)$$

These two options have equal utility when $p_1 = \frac{3}{5}$.

Thus, a Nash equilibrium for this game is $(p_1, p_2) = (\frac{3}{5}, \frac{2}{5})$.

**Example**

$\ell + 1$ bidders all value an object at $v$. Whoever bids highest pays their bid and gets the object, and everyone else still pays their bid.

Clearly, any bid above $v$ is strictly dominated. Consider a symmetric Nash equilibrium in which every player bids in $[b_*, v]$ with CDF $G(b)$. The payoff to a player bidding $b$ is

$$u(b) = (v - b)G(b)^\ell - b(1 - G(b)^\ell) = vG(b)^\ell - b$$

We know $u(v) = 0$, and for this to be a Nash equilibrium, this utility must be constant through the interval $[b_*, v]$. Thus

$$G(b) = \left(\frac{b}{v}\right)^{1/\ell} \qquad \forall b \in [b_*, v]$$

For this to be a valid CDF, we need $b_* = 0$.

**Example**

Consider a Bertrand oligopoly game in which two firms compete for $N$ customers, each willing to pay at most $p^*$ for each unit of the product. The firms can produce no more than $\bar{q}$ items, and need to decide what price to post.

$$u_1(p_1, p_2) = \begin{cases} p_1\bar{q} & \text{if } p_1 < p_2 \leq p^* \\ p_1(N - \bar{q}) & \text{if } p^* > p_1 > p_2 \\ \frac{1}{2}p_1 N & \text{if } p^* > p_2 = p_2 \end{cases}$$

- If $\bar{q} < 50$, both firms post the monopoly price $p^*$.
- If $\bar{q} > 100$, there is a single Nash equilibrium at $p_1 = p_2 = 0$.
- If $\bar{q} \in (50, 100)$, there cannot be a pure-strategy Nash equilibrium (if $p_1 = p_2 > 0$, one of the firms would deviate slightly and get all the profit, if $p_1 = p_2 = 0$, one of the firms can go slightly higher and make the profit in excess of the other firm's capacity, and if $p_1 < p_2$, firm 1 has no reason not to increase up to $p_2$).

  Assume the mixed (symmetric) equilibrium is for each firm to set the price somewhere in $[\rho, p^*]$, with CDF $G(p)$. Then if firm 1 bids $p$, its payoff is

  $$u_1(p) = p\bar{q}(1 - G(p)) + p(N - \bar{q})G(p)$$

  This gives

  $$G(p) = \frac{p\bar{q} - K}{p(2\bar{q} - N)}$$

  Using $G(p^*) = 1$ and $G(\rho) = 0$ allows us to find $K$ and $\rho$. We then reason that the value outside that range is less advantageous.

## 2.4 Bayesian Nash Equilibrium

In a Bayesian came, each player $i$ has a payoff function $u_i(s_i, s_{-i}, \theta_i)$ where $\theta_i \in \Theta_i$ is a random variable chosen by nature and observed only by player $i$. The joint probability of the $\theta_i$'s is given by $F(\theta_1, \cdots, \theta_\ell)$, which is assumed to be common knowledge among the players. Letting $\Theta = \Theta_1 \times \cdot \times \Theta_\ell$, a Bayesian game is summarized by the data

$$[\ell, \{\mathcal{S}_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$$

A pure strategy for player $i$ is a function $s_i : \Theta_i \to \mathcal{S}_i$ (a *decision rule*). Player $i$'s pure strategy set $\mathscr{S}_i$ is therefore the set of all such functions.

Player $i$'s expected payoff given a profile $(s_1(\cdot), \cdots, s_\ell(\cdot))$ is then given by

$$\tilde{u}_i(s_1(\cdot), \cdots, s_\ell(\cdot)) = \mathbb{E}_\theta\left[u_i(s_1(\theta_1), \cdots, s_\ell(\theta_\ell), \theta_i)\right]$$

**Definition 10. (Pure strategy Bayesian Nash equilibrium)** A pure strategy Bayesian Nash equilibrium is a profile of decision rules $(s_1(\cdot), \cdots, s_\ell(\cdot))$ that constitutes a Nash equilibrium of the game $\Gamma_N = [\ell, \{\mathscr{S}_i\}, \{\tilde{u}_i(\cdot)\}]$. That is, for every $i$

$$\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \tilde{u}_i(s_i'(\cdot), s_{-i}(\cdot))$$

for every $s_i'(\cdot) \in \mathscr{S}_i$.

**Theorem 4.** A profile of decision rules $(s_1(\cdot), \cdots, s_\ell(\cdot))$ is a Bayesian Nash equilibrium if and only if for all $i$ and all $\tilde{\theta}_i \in \Theta_i$ occurring with positive probability,

$$\mathbb{E}_{\theta_{-i}}\left[u_i(s_i(\tilde{\theta}_i), s_{-i}(\theta_{-i}), \tilde{\theta}_i)|\tilde{\theta}_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i(s_i', s_{-i}(\theta_{-i}), \tilde{\theta}_i)|\tilde{\theta}_i\right]$$

for all $s_i' \in \mathcal{S}_i$.

This theorem implies that we can effectively consider each 'type' of player $i$ (corresponding to a different $\theta_i \in \Theta_i$ as a separate player, who maximizes his expected payoff (with expectation taken over all the types of his opponents) given his type.

**Example**

Consider a first-price sealed bid auction with $\ell+1$ bidders, where each player's valuation is drawn from a CDF $F$ with support $[v_{\min}, v^{\max}]$.

We will show there is a pure-strategy Nash equilibrium in which each bidder bids $s(v_i)$, where $v_i$ is the bidder's valuation.

If bidder $i$ bids $b$, his utility is

$$\begin{aligned} u_i(b) &= \mathbb{P}\left[b > \max_{k\neq i} s(v_k)\right](v_i - b) \\ &= F\left[s^{-1}(b)\right]^{\ell}(v_i - b) \end{aligned}$$

To make algebra simpler, let $r$ be such that $b = s(r)$. We then have

$$u_i = F(r)^{\ell}(v_i - s(r))$$

At the optimal $b$ (or, equivalently, the optimal $r$)

$$\frac{\mathrm{d}u_i}{\mathrm{d}r} = \ell F(r)^{\ell-1} f(r)(v_i - s(r)) - F(r)^{\ell} s'(r) = 0$$

His optimal bid, $b_*$ satisfies

$$u_i'(b_*) = \ell F\left[s^{-1}(b)\right]^{\ell-1} f\left[s^{-1}(b)\right]^{\ell-1}$$

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### 2.5 Mistakes & Trembling Hand Perfection

We now consider the possibility that some players may make mistakes. In particular, that there is a minimum probability $\epsilon_i(s_i) \in (0,1)$, with $\sum_{s_i\in\mathcal{S}_i}\epsilon_i(s_i) < 1$, that the player will play each of his strategies $s_i$.

We model this by defining a *perturbed strategy set* for the game

$$\Delta_\epsilon(\mathcal{S}_i) = \left\{\sigma_i : \sigma_i(s_i) \geq \epsilon_i(s_i), \sum_{s_i\in\mathcal{S}_i}\sigma_i(s_i) = 1\right\}$$

**Definition 11. ((Normal-form) trembling-hand perfect Nash equilibrium)** A Nash equilibrium $\sigma$ in a game $\Gamma_N$ is trembling-hand perfect if there is some sequence of perturbed games $\{\Gamma_{\epsilon^k}\}_{k=1}^{\infty}$ that converges to $\Gamma_N$ (in the sense that $\lim_{k\to\infty}\epsilon_i^k(s_i) = 0$ for all $i$ and $s_i \in \mathcal{S}_i$), for which there is *some* associated sequence of Nash equilibria $\{\sigma^k\}_{k=1}^{\infty}$ that converges to $\sigma$.

This definition is not as strong as it could be – indeed, note that it only requires *some* perturbed games to have equilibria close to $\sigma$, not all.

A more manageable way to test for this concept is provided in the following theorem

**Theorem 5.** A Nash equilibrium of a game is normal form trembling-hand perfect if and only if there is some sequence of totally mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}\sigma^k = \sigma$ and $\sigma_i$ is a best response to every element of sequence $\{\sigma_{-i}^k\}_{k=1}^{\infty}$ for all $i$.

And a direct consequence...

**Theorem 6.** If $\sigma$ is a (normal form) trembling-hand perfect Nash equilibrium, then $\sigma_i$ is not a weakly dominated strategy for any $i$. Hence, in any such equilibrium, no weakly dominated pure strategy can be played with positive probability.

*Note*: The converse is, generally, only true in two-person games. Thus, this concept narrows strategies down further.

## 3 Dynamic Games

We now consider a situation in which sequentiality in the plays may result in the Nash equilibrium being untenable.

### 3.1 Sequential Rationality, Backward Induction, and Subgame Perfection

**Definition 12. (Subgame)** A *subgame* of an extensive form game $\Gamma_E$ is a subset of the game having the following properties:

- It begins with an information set containing a single node, all decision nodes that are this node's successors and *only* those nodes.
- If node $x$ is in the subgame, then every node in $x$'s information set is also in the subgame.

**Definition 13. (Subgame perfect Nash equilibrium)** A profile of strategies $\sigma$ is a *subgame perfect Nash equilibrium* (SPNE) if it induces a Nash equilibrium in every subgame of $\Gamma_E$.

It is particularly interesting to note that in finite games of perfect information, every new node begins a new game. As such, the SPNEs in such a game can be found by *backwards induction* – starting at the leaves of the tree and finding Nash equilibria at each decision. This is called *backward induction*.

**Example**

Consider a $T$ period game. In odd periods, player 1 makes an offer $x$. If player 2 accepts this offer, the payoffs are $\left(\delta^{t-1}x, \delta^{t-1}(1-x)\right)$ with $\delta < 1$. If player 2 rejects the offer, the game moves on and the roles are reversed. If period $T$ is reached without an acceptance, the payoff is $(0,0)$.

Suppose player 1 makes the final offer in period $T$ (ie: $T$ is odd):

**Period** $T$ – **player 1** – Player 1's offer will be $(1,0)$, which player 2 will take, with payoffs $(\delta^{T-1}, 0)$.

**Period** $T-1$ – **player 2** – Player 2 wants player 1 to accept (else, her payoff will be 0 as above). She'll therefore make an offer giving a payoff of at least $\delta^{T-1}$ to player 1. This offer is $\left(\frac{\delta^{T-1}}{\delta^{T-2}}, 1-\frac{\delta^{T-1}}{\delta^{T-2}}\right)$ which gives payoffs

$$(\delta^{T-1}, \delta^{T-2} - \delta^{T-1})$$

**Pariod** $T-2$ – **player 1** – For player 2 to accept, player 1 needs to offer her at least $\delta^{T-2} - \delta^{T-1}$. This offer is $\left(1 - \frac{\delta^{T-2}-\delta^{T-1}}{\delta^{T-3}}, \frac{\delta^{T-2}-\delta^{T-1}}{\delta^{T-3}}\right)$, giving payoffs

$$(\delta^{T-3} - \delta^{T-2} + \delta^{T-2}, \delta^{T-1} - \delta^{T-1})$$

This outcome for player 1 is greater than $\delta^{T-1}$ (which he'd have got if 2 had rejected his offer and the step above had been reached). Thus, it is in his interest to make this proposal and get 2 to accept.

**Period** $T-3$ – **player 2** – using similar logic, player 2 would have to propose a number leading to a payoff

$$(\delta^{T-3} - \delta^{T-2} + \delta^{T-1}, \delta^{T-4} - \delta^{T-3} + \delta^{T-2} - \delta^{T-1})$$

Continuing these steps, we find a payoff of

$$\left(\sum_{t=0}^{T-1}(-\delta)^t, \sum_{t=1}^{T-1}(-\delta)^t\right)$$

or, bearing in mind $T$ is odd,

$$\left(\frac{1+\delta^T}{1+\delta}, \frac{\delta^T - \delta}{1+\delta}\right)$$

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*Generalized backwards induction* finds SPNEs in more generate finite dynamic games. It simply requires us to start by finding Nash equilibria for each of the *final* subgames, and then to work our way up from there. If the subgames contain multiple Nash equilibria, then each needs to be considered separately.

The following theorem show that strategies obtained using this method coincide with SPNEs for games of incomplete information.

> **Theorem 7.** Let $S$ be a subgame of $\Gamma_E$ and let $\sigma^S$ be an SPNE of $S$. Consider the *reduced game*, $\hat{\Gamma}_E$ obtained from $\Gamma_E$ by replacing the subgame $S$ with a terminal node whose payoffs are those induced by $\sigma^S$. Then
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> - If $\sigma$ is an SPNE of $\Gamma_E$ in which $\sigma_{\text{Subgame } S}$, then $\sigma_{\text{Outside } S}$ must form an SPNE for $\hat{\Gamma}_E$.
> - I f $\hat{\sigma}$ is an SPNE of $\hat{\Gamma}_E$, then $\sigma = (\hat{\sigma}, \sigma^S)$ is an SPNE of $\Gamma_E$.

*Proof.*

- Suppose there is a deviation from $\sigma_{\text{Outside } S}$ in $\hat{\Gamma}_E$ that makes a player better off. That same deviation in $\sigma$ (leaving all moves in $S$ unchanged) would make the player better off in $\Gamma_E$ contradicting the fact $\sigma$ is an SPNE for $\Gamma_E$.
- We need to show that $\sigma$ induces an SPNE on every subgame $S'$ of $\Gamma_E$. If $S \notin S'$, there is nothing to show.

  Otherwise, suppose there is a deviation from $\sigma_{\text{Subgame } S'}$ that makes the player better off in $S'$. Then since $\sigma_S$ an an SPNE for $S$, that deviation must involve moves outside $S$ and in $S'$. Thus, $\hat{\sigma}$ cannot be an SPNE of $\hat{\Gamma}_E$.

As required. □

The following theorem concerns a specific kind of game

**Theorem 8.** Consider an $\ell$-player extensive form game involving successive plays of $T$ simultaneous move games $\Gamma_N^t = \left[\ell, \{\Delta(S_i^t)\}, \{u_i^t(\cdot)\}\right]$, with the players observing the pure strategies played in each game immediately after conclusion of play, and in which the payoff is equal to the sum of the payoffs in all $T$ games. If there is a **unique** Nash equilibrium $\sigma^t$ in each game $\Gamma_N^t$, then there is a unique SPNE for the full game, which consists of playing those strategies in each game regardless of what has happened previously.

*Proof.* Clearly, this is true for $T = 1$ (if we only have one game). Now, suppose this is true for $T = n-1$, and consider a situation with $T = n$. After we've played game 1, we must play an SPNE of the resulting $n - 1$ games (since these games form a subgame), and by the inductive hypothesis, this SPNE involves playing the Nash equilibrium in each case. □

## 4 Beliefs and Sequential Rationality

This concept is unfortunately not so useful in games without proper subgames. This occurs, for example, if there is uncertainty at every step of the game. In such cases, we require moves to be rational with respect to *some belief* of how the uncertainty was resolved.

**Definition 14. (System of Beliefs)** A *system of beliefs* $\mu$ in an extensive form game $\Gamma_E$ is a specification $\mu(x) \in [0,1]$ for each decision node $x$ in $\Gamma_E$ such that

$$\sum_{x \in H} \mu(x) = 1$$

For all information sets $H$ in the game.

This represents the probability a particular player will be at a certain point in an uncertainty set conditional on having reached that uncertainty set.

We also let $\mathbb{E}[u_i | H, \mu, \sigma_i, \sigma_{-i}]$ denote player $i$'s expected utility starting at his information set $H$, assuming the conditional probabilities of being at the various nodes in $H$ are given by $\mu$, that he plays strategy $\sigma_i$ and that his rivals uses strategies $\sigma_{-i}$.

**Definition 15. (Sequentially rational strategy profile)** A strategy profile $\sigma$ in the extensive form game $\Gamma_E$ is *sequentially rational* at information set $H$ given a set of beliefs $\mu$ if, denoting by $\iota(H)$ the player who moves at information set $H$, we have

$$\begin{aligned} &\mathbb{E}\left[u_{\iota(H)} | H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}\right] \\ &\qquad \geq \mathbb{E}\left[u_{\iota(H)} | H, \mu, \bar{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}\right] \end{aligned}$$

In other words $\sigma$ is sequentially rational if no player finds it rational to revise his strategy (once his information set has been reached) given his beliefs about what has already occurred (as embodied in $\mu$).

**Definition 16. (Weak perfect Bayesian equilibrium)** A profile of strategies and a system of beliefs $(\sigma, \mu)$ is a *weak perfect Bayesian equilibrium* (weak PBE) in an extensive form game $\Gamma_E$ if it has the following properties

- The strategy profile $\sigma$ is sequentially rational given belief system $\mu$.
- The system of beliefs $\mu$ is derived from $\sigma$ through Bayes' rule, whenever possible. In other words, for any information set $H$ such that $\mathbb{P}(H|\sigma) > 0$ (ie: any information set that may be reached with positive probability under $\sigma$), we have

$$\mu(x) = \frac{\mathbb{P}(x|\sigma)}{\mathbb{P}(H|\sigma)} \qquad \forall x \in H$$

**EXAMPLE**

Consider the following game

| | Fight | Accomodate |
|---|---|---|
| Out | (0,2) | (0,2) |
| In1 | $(-1,-1)$ | $(3,-2)$ |
| In2 | $(\gamma, -1)$ | (2,1) |

In which player 1 first makes a decision, followed by player 2. If player 1 picks 'out', player 2 observes it. If player 1 picks one of the 'in' strategies, player 2 does not know which.

Consider two possibilities.

$$\boxed{\gamma > 0}$$

In this case, 'out' is strictly dominated by 'In2'. As such, player 1's strategy is $\sigma^1 = (0, p_1, 1-p_1)$.

Player 2's uncertainty set is reached with probability 1, and his beliefs are $\mu(\text{In1}) = p_1$ and $\mu(\text{In2}) = 1 - p_1$.

$$\boxed{\gamma \in (-\tfrac{2}{3}, 0)}$$

Consider player 2 – if player 1 picks 'out', she is indifferent between F and A. If not, let player 2's beliefs be $\mu(\text{In1}) = p$ and $\mu(\text{In2}) = 1-p$. Her payoffs if she picks $F$ or $A$ are then

$$\mathbb{E}[u_2|F,\mu] = -1 \qquad \mathbb{E}[u_2|A,\mu] = 1 - 3p$$

As such, if $p > \frac{2}{3}$, player 2 chooses F.

We therefore have our first weak PBE – (Out, F), with $\mu(\text{In1}) \geq \frac{2}{3}$. (Note that since player 2's information set is never reached, $\mu$ need not conform to Bayes' rule).

Suppose, instead, that player 1 decides to enter the game. He then faces the following subgame

| | Fight | Accomodate |
|---|---|---|
| In1 | (−1,−1) | (3,−2) |
| In2 | ($\gamma$, −1) | (2,1) |

This has NE

$$\sigma^1 = \left(\tfrac{2}{3}, \tfrac{1}{3}\right) \qquad \sigma^2 = \left(\tfrac{1}{2+\gamma}, \tfrac{1+\gamma}{2+\gamma}\right)$$

Now consider; the payoff to player 1 assuming he always enters the game and follows the strategy above is

$$-\left(\tfrac{2}{3}\tfrac{1}{2+\gamma}\right) + \gamma\left(\tfrac{1}{3}\tfrac{1}{2+\gamma}\right) + 3\left(\tfrac{2}{3}\tfrac{1+\gamma}{2+\gamma}\right) + 2\left(\tfrac{1}{3}\tfrac{1+\gamma}{2+\gamma}\right)$$
$$= \tfrac{3\gamma+2}{2+\gamma}$$

This is $> 0$, since $\gamma > -\frac{2}{3}$. Thus, $(\sigma^1, \sigma^2)$ is a weak PBE, with $\mu = \sigma^1$.

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**Theorem 9.** A strategy profile $\sigma$ is a Nash equilibrium of the extensive form game $\Gamma_E$ if and only if there exists a system of beliefs $\mu$ such that

- The strategy profile $\sigma$ is sequentially rational given belief system $\mu$ <u>at all information sets $H$ such that $\mathbb{P}(H|\sigma) > 0$.</u>
- The system of beliefs $\mu$ is derived from strategy profile $\sigma$ through Bayes' rule whenever possible.

Notice that the only difference between a Nash equilibrium and a weak PBE is the underlined phrase above.

**Definition 17. (Sequential equilibrium)** A strategy profile and system of beliefs $\{\sigma, \mu\}$ is a *sequential equilibrium* of extensive form game $\Gamma_E$ if it has the following properties

- Strategy profile $\sigma$ is sequentially rational given belief system $\mu$.
- There exists a sequence of completely mixed strategies $\{\sigma^{(k)}\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \sigma^{(k)} = \sigma$ such that $\mu = \lim_{k\to\infty} \mu^{(k)}$, where $\mu^{(k)}$ denotes the beliefs derived from strategy profile $\sigma^{(k)}$ using Bayes' rule.

In some sense, this kind of equilibrium requires beliefs to be justifiable under the assumption other players may be small 'mistakes' in their optimal strategies.

## 5 Repeated games

A repeated game consists of repetitions of finite-move simultaneous games of one period, $\{\ell, \{A_i\}_{i\in\ell}, u\}$. We assume every player observes the actions chosen by all other players at the end of each period.

**Definition 18. (Strategy)** Let $h_t = (a^1, \cdots, a^{t-1})$ be a history of length $t$, where $a^\tau$ is the action profile played at time $\tau$ in the game. Let $H_t$ be the set of histories of length $t$, and $\mathscr{H} = \cup_{t\geq 0} H_t$ denote the set of histories of the game.

A *pure behavioural strategy* is

$$s_i : \mathscr{H} \to A_i$$

A *mixed behavioural strategy* is

$$\sigma_i : \mathscr{H} \to \Delta(A_i)$$

A strategy profile generates a sequence of payoffs for each player in each period. Players discount the future at the common factor $\delta < 1$. Thus, for mixed strategies

$$U_i(\sigma) = (1-\delta)\sum_{t=1}^{T} \delta^{t-1}\left(\sum_{h_t \in H_t} \pi(h_t|\sigma)u_i(\sigma(h_t))\right)$$

where $\pi(h_t|\sigma)$ is the probability of realizing history $h_t$ given that the players are following strategy profile $\sigma$.

Every history $h_t$ can be thought of the initial node in a subgame.

## 5.1 One-short deviation principle

Let $\sigma_i|h_t$ be the restriction of strategy $\sigma_i$ to the subgame following history $h_t$.

**Definition 19. (Profitable one-short deviation)** Let $\sigma$ be given. A *profitable one-shot deviation* for player $i$ is a strategy $\sigma_i' \neq \sigma_i$ such that

- $\sigma_i(h_t) = \sigma_i'(h_t)$ for all histories except for one particular history $h_{t'}$ (in other words, player $i$ only deviates for one particular history).
- $U_i(\sigma_i'|h_{t'}, \sigma_{-i}|h_{t'}) > U_i(\sigma_i|h_{t'}, \sigma_{-i}|h_{t'})$

A Nash equilibrium can have a profitable one-shot deviation as $h_{t'}$ may not be on the equilibrium path. An SPNE cannot.

**Theorem 10.** A strategy $\sigma$ is an SPNE if and only if there are no profitable one-shot deviation.

**Theorem 11.** Suppose that $\sigma$ is a strategy profile of the repeated game with the following property

$$\sigma(h_t) = \gamma(t) \qquad \forall h_t \in H_t$$

Then $\sigma$ is an SPNE if and only if $\gamma(t)$ is a Nash equilibrium of the stage game for all $t$.

## 5.2 Finitely Repeated Games

**Theorem 12. (I)** f the stage game has a unique Nash equilibrium $\alpha^*$, the unique SPNE of the finitely repeated game is $\sigma^*(h_t) = \alpha^*$, for all $h_t$ and $t$.

As such, repeating a game with a unique Nash equilibrium does not help in fostering cooperation.

**Example**

Consider the following game

| | L | C | R |
|---|---|---|---|
| S | (10,10) | (2,8) | (0, 13) |
| M | (8,2) | (5,5) | (0, 3) |
| B | (13,0) | (3,0) | (1,1) |

(M,C) and (B,R) are Nash equilibria. S and L are strictly dominated, even though they can be thought of as the ‘cooperative’ outcome.

By Theorem 11, any strategy that alternates between (M,C) and (B,R) is an SPNE. We can also, however, foster cooperation.

$$\boxed{\delta = 1}$$

$$\sigma_1(h_t) = \begin{cases} S & \text{if } t = 1 \\ S & \text{if } t < T \text{ and } a^\tau = \text{(S,L)} \ \forall \tau < t \\ B & \text{if } t < T \text{ otherwse} \\ M & \text{if } T = T \text{ and } a^\tau = \text{(S,L)} \ \forall \tau < t \\ B & \text{if } T = T \text{ otherwise} \end{cases}$$

and

$$\sigma_1(h_t) = \begin{cases} L & \text{if } t = 1 \\ L & \text{if } t < T \text{ and } a^\tau = \text{(S,L)} \ \forall \tau < t \\ R & \text{if } t < T \text{ otherwse} \\ C & \text{if } T = T \text{ and } a^\tau = \text{(S,L)} \ \forall \tau < t \\ R & \text{if } T = T \text{ otherwise} \end{cases}$$

(In short – play (S,L) until the other player deviates, at which point start playing (B,R). If the other player hasn’t deviated by the last period, play (M,C).

Now consider – of the equilibrium path, the players are playing the NE (B,R). We therefore only need to check that there are no profitable one-shot deviations from the equilibrium path. The most profitable such deviation will be in period $T-1$ (when the player has already gained as much as possible from cooperation).

Player 1 may deviate there by playing the best response to L – ie: by playing B. It would then earn 13 in the period, but then only 1 in the next period (when (B,R) will be played), thus resulting in a gain of 14. This is less than the gain of 10 + 5 = 14 that it would get should it continue to cooperate.

$$\boxed{\delta < 1}$$

If $\delta < 1$, the reasoning above only applies if

$$13 + \delta < 10 + 5\delta \Rightarrow \delta > \frac{3}{4}$$

Otherwise, consider a strategy identical to the one above *except* that the play of (M,C) also extends to period $T-1$. So the strategy becomes to play (S,L) until the other player deviates, after which point (B,R) should be played. If the other player hasn't deviated by time $T-1$, play (M,C) at times $T-1$ and $T$.

Now, it no longer makes sense to deviate at $T-1$. The most profitable deviation is now at $T-2$ (when the most cooperation has been encouraged thus far). Player 2 will be playing L. Suppose player 1 were to play his best response (B) and get 13. Play would then immediately switch to (B,R) and the payoff to player 1 would be

$$13 + \delta + \delta^2$$

as compared to $10 + 5\delta + 5\delta^2$. The later is less than the latter for all $\delta > \frac{1}{2}$.

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The takeaways from the example above are:

- Fostering cooperation is possible when there are multiple NEs, ranked in terms of Pareto desirability.
- At the terminal phase of the game, the best NE must be played.
- Earlier in the game, the cooperative outcome is played. The worse NE is used to threaten the other player and discourage deviations.
- The smaller $\delta$, the shorter the cooperative period.

### 5.3 Infinitely repeated games