B6015 – Decision Models
Review Session 2

The primary aims of this review session are:

- To review cash-flow models, as introduced in class.
- To review the mechanics of sensitivity reports, as introduced in class.
- To thoroughly practice both of the above in Excel.

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You are a trustee of the Bolumbia Business School, located in the Eveningside Heights neighborhood of Gotham. The school has long-term plans to move to a new location, further uptown, and expects the project – from start to finish – to last a total of 11 years. The trustees have estimated the costs of the project, and expect the following moneys will be required

- The main project will require $500,000 in its first year (year 1), and this requirement will increase thereafter at a rate of 11.5% per year for the following 10 years (years 2 to 11).
- A side project, beginning at the start of year 5, will require $600,000 in its first year, and this requirement will also increase thereafter at a rate of 11.5% per year for the remaining 6 years (years 6 to 11).

Assume that all funds for a given year are needed on the first day of the year.

In these tough economic times, the trustees resolve to raise all the money they will need before the project even begins on the first day of year 1. When that day arrives, the trustees have a choice as to what to do with money they are holding for future years. They can either decide to place the money in a money market fund yielding 5% per year, or they can invest in a number of AAA-rated bonds issued by three different companies. Table 1 summarizes the provisions of these bonds, as well as the times when they are available for purchase. Money can be invested in (or withdrawn from) the money market fund at the start of each year. Due to internal policies, the trustees are not able to purchase more than 4,000 company B bonds in total. Money can be added to or withdrawn from the money-market account at any time with no penalty, but bonds cannot be shorted. Assume also that you can invest any dollar amount to buy any (even fractional) number of bonds.

The table should be interpreted as follows: at the start of year 1,

	Available at start of year	Maturity	Coupon payment	Price	Par value
Company A	1	10 years	70	834.32	1,000
Company B	1,2	8 years	100	945.73	1,000
Company C	1,4	5 years	75	873.30	1,000

Table 1: Summary of available bonds. All prices are in dollars.

the trustees may choose to buy bonds issued by Company A for \$834.32 per bond. Each bond will pay the holder \$70 per year at the start of each of the next 9 years (years 2 – 10) and then \$1,070 (par value plus the last coupon payment) at the start of year 11. Furthermore, the start of year 1 is the only opportunity the trustees will have to purchase Company A bonds. Similar interpretations apply to the other bonds.

Part A

How much money do the trustees need to raise before the start of year 1, and how should they manage that money during the 11 years of the construction project?

Solution

Let's begin by going through the three steps introduced in review session 1 to formulate this problem.

Decision variables : in this problem, what decisions do we need to make? Clearly, the trustees need to decide how much money to raise and how much to invest in each of the various bonds. As I noted in review session 1, however, this isn't precise enough – we need to specify these decisions more accurately.

The first and most obvious decision the trustees will need to make is how much to invest in each of the bonds when they are offered. Since bonds are offered at five different times, this leads to five decision variables[1]:

[1]We will carry out similar scaling to that we carried out in review session 1, in which we express decision variables in thousands.

- A_1, the number of thousand of company A bonds to buy in year 1.
- B_1, the number of thousand of company B bonds to buy in year 1.
- B_2, the number of thousand of company B bonds to buy in year 2.
- C_1, the number of thousand of company C bonds to buy in year 1.

- C_4, the number of thousand of company C bonds to buy in year 4.

The next decision the trustees need to make is how much to invest in the money-market fund during each given year:

- I_1, the amount in thousands of dollars in the fund at the start of year 1, to be withdrawn at the start of year 2[2].
- I_2, the amount in thousands of dollars in the fund at the start of year 2, to be withdrawn at the start of year 3.
- etc...
- I_{10}, the amount in thousands of dollars in the fund at the start of year 10, to be withdrawn at the start of year 11.

And that's it! You might be tempted to define an additional decision variable describing the total amount of money the trustees will need to raise at the start of year 1. However, this variable would be superfluous – indeed, if you know how much the trustees intend to invest in each bond, and how much they intend to invest in the money-market fund, you can easily calculate the amount they will need to raise to meet these aims. Thus, it's not a 'decision' in its own right. [3]

Objective function : in this problem, the objective is clearly to minimize the amount of money the trustees initially need to raise. This is given (in thousands of dollars) by

$$I_1 + 834.32A_1 + 945.73B_1 + 873.30C_1 + 500$$

To see why, consider that the moneys in the first year are required for

- Investing in the year 1 money-market account (given by I_1).
- Buying year-1 bonds (the price of year-1 bonds purchased is given by $834.32A_1 + 945.73B_1 + 873.30C_1$).
- The $500,000 needed for the project in year 1.

Contraints : we now consider the various constraints in this problem

- The first constraint is that on company B bonds – we can buy no more than 4,000 of them. This is easy to write in terms of our decision variables

$$B_1 + B_2 \leq 4$$

[2] A number of people found this concept confusing during the review session. Surely there should be an option to *leave* some money in the money market account – why should we have to withdrawn all of the money invested at the start of year 1 once we get to the start of year 2? The answer is that of course we can leave some money in the money-market account. However, any money we keep in the money market account will then have to be accounted for in I_2, the variable that represents the amount in the money market account at the start of year 2. Thus, conceptually, I refer to 'taking the money out of I_1' and 'putting it into I_2', even though really, the money's just staying in the money market account, and just being supplemented or depleted as appropriate.

[3] Note, however, that an alternative formulation is possible in which you *do* include a decision variable for the total amount raised, but *no* decision variable for the total amount invested in the money-market fund at the end of each year. In this case, knowing how much money is available at the start and how much is invested in each bond, it is easy to work out how much money is left over at the end of each year, and therefore how much money can will be invested in the money-market fund. In a sense, this alternative formulation is more frugal – it replaces 10 decision variables with a single decision variable. However, the formulation we use here happens to be slightly more convenient to implement in Excel. If you want some practice, try implementing the alternative formulation – it's not too hard – and ask me if you'd like some tips. I've included the alternative formulation as a tab in the Excel solution spreadsheet for this review session.

- Our second constraint is also our most complex one – we need to ensure that every year, the amount of money we use is equal to the amount of money we have

$$\text{Money we have} = \text{Money we use}$$

Let's consider how this plays out at the end of year 1/start of year 2

The money we have comes from

  - Coupon payments from bonds purchased at the start of year 1. Company A, B and C bonds pay coupons \$70, \$90 and \$75 respectively, and so this is given by

$$70A_1 + 100B_1 + 75C_1$$

  - Proceeds from any money invested in money-market account in year 1. Since the interest rate is 5%, this is given by

$$1.05I_1$$

As such, the total money we have is

$$70A_1 + 90B_1 + 75C_1 + 1.05I_1$$

The money we need goes to

  - The money we need for our project in year 2 (payable at the start of year 2) is \$ 557,500 (\$500,000 increased by 11.5%).
  - The money we need to buy bonds at the start of year 2. Since only company B offers bonds at the start of year 2, the money we need for this is

$$945.73B_2$$

  - The money we invest in the money-market account at the start of year 2, given by I_2.

As such, the total money we need is

$$945.73B_2 + I_2 + 557.50$$

Thus, the constraint for the end of year1/start of year 2 is given by

$$70A_1 + 100B_1 + 75C_1 + 1.05I_1 = 945.73B_2 + I_2 + 557.50$$

Similar constraints apply to all other years. For the sake of brevity, I won't list them here, but make sure you do in an exam! (Typically, an exam question would include many fewer years, thus making the problem less cumbersome to write down).

The last constraint, however, for end of year 10/start of year 11, will be slightly different. Instead of making the cash flows *equal* to the cash required, we will require the cash flows to be *greater than or equal to* the cash flows required. This is because the problem might otherwise be infeasible – indeed, it'd possible that the particular combination of bonds given simply doesn't provide enough flexibility to achieve the right cash flows *every* year. Thus, by requiring that the last cash flow be only greater than or equal to the requirement, we give ourselves some slack. This is not required in years 1 to 10 because in those year, we have a money-market account that can pick up any slack – if the bonds provide us with more money than we need, we can just invest that money in the money-market account. In the last year, we don't have that flexibility.

- Finally, we require that bonds cannot be shorted – thus,

$$A_1, B_1, B_2, C_1, C_4 \geq 0$$

See the Excel spreadsheet for a solved model.

Part B

Without resolving your model, answer the following questions:

1. As mentioned above, financial regulations require that no more than 4000 bonds be purchased from company B in total. If this constraint were to be relaxed to require that no more than 5000 company B bonds be purchased in total, how much money would the trustees now need to raise in year 1?

2. You should have found, in Part A, that your optimal solution prescribes that about 2600 Company A bonds should be bought in year 1. Imagine the government now subsidizes these bonds – for every bond bought, a fixed rebate is given. How high would this rebate need to be for the optimal money management policy to change in favor of buying more Company A bonds?

3. In the current model of part A, the trustees need to plan for a cash requirement of \$1,706,700 in year 7. Suppose that there will in fact be another one-off \$15,000 requirement in

that year. How much money would the trustees now need to raise in year 1?

Solution

This part of the question requires a sensitivity report (included in the solutions).

Before we actually consider the particular problem in the review session, let's review the basics of sensitivity reports. We'll do this in the context of the following much simpler (albeit contrived!) optimization problem with two decision variables, x and y:

- Let x be the number of hours you spend tutoring each week, and y be the amount of time you spend bartending.
- Tutoring gets you \$2 an hour, and bartending get you \$1 an hour. Your objective is to increase $2x + y$, the total money you'll make.
- The total amount of time you have per week is 4 hours, so $x + y \leq 4$.
- For tax reasons, you don't want to make more than \$5 in total, so $2x + y \leq 5$.
- Tutoring requires three hours of preparation beforehand, whereas bartending only takes one hour of preparation. You want to spend no more than 7 hours total per week preparing. Thus, $3x + y \leq 7$.

Thus, our optimization problem is

$$\begin{aligned} \max \quad & 2x + y && (1) \\ \text{s.t.} \quad & x + y \leq 4 && (2) \\ & 2x + y \leq 5 && (3) \\ & 3x + y \leq 7 && (4) \\ & x \geq 0, y \geq 0 \end{aligned}$$

This problem is included in the solution Excel spreadsheet for this review session, under the "Intro to sensitivity" tab. I have included the sensitivity report for this problem there. There are two parts to the sensitivity report. One deals with the *constraints* in the problem (lines 2, 3 and 4 above), and the other deals with the objective function (line 1 above). We will consider each in turn

Constraints : the *bottom* part of the sensitivity report deals with constraints. The key quantity here is the *shadow price* – each constraint has a shadow price, and the shadow price answers the following question

> If I were to *increase* the right-hand-side of this constraint by 1, how much would my objective

function change by?

For example, in the example above, the shadow price of the second constraint (line 3) is 1. What this means is that if we were to increase the RHS (right-hand-side) of this constraint by 0.5, our objective function would increase by $0.5 \times 1 = \$0.5$. Similarly, if we were to *decrease* the RHS of this constraint by 0.5, our objective would drop by this amount.

In practical terms, this says something about the second constraint. Namely, that if our tax constraints became more relaxed, we'd be able to make more money.

However, it's clear that this won't be true indefinitely. Clearly, if we were to relax this tax constraint by an enormous amount, the solution itself wouldn't increase by an enormous amount – something else (maybe the total amount of time available) would kick in and restrict our profit. Indeed, shadow prices are not valid indefinitely. This is where the "allowable increase" and "allowable decrease" columns come in for each constraint. They tell us by how much the right-hand-side can be increased or decreased while keeping the shadow price valid.

In the case of the tax constraint, the allowable increase in 0.5 and the allowable decrease is 1. This means that the right-hand-side of the constraint (currently 5) can increase as far as 5.5, and decrease as low as 4, while keeping the shadow price given valid. Past this range, the shadow price is no longer valid[4].

Other shadow prices work in similar ways.

Objective function : the *middle part* of the sensitivity report deals with the objective function. The key quantity here are the allowable increases and decreases. These quantities allow us to answer the following question:

> By how much can I change a particular objective function coefficient before my optimal solution changes?

This concept can be a bit abstract, so let's apply it to our particular problem. Consider the x variable (number of hours of tutoring)

- The allowable increase is 1E-07. This should be read as 1×10^{-7}, which is a very, very small number indeed – so effectively 0.[5]
- The allowable decrease is 1.

[4]This does not mean, however, that we can't say anything about what would happen if we were to change the RHS outside that range. Indeed, *relaxing* a constraint can only *improve* our optimal solution, and vice-versa for tightening a constraint. See review session 1, where we used this logic.

[5]If, on the other hand, the number had been 1E07, without the minus sign, this would have meant 1×10^7, a huge – effectively infinite – number. The minus sign is important.

This means that the coefficient in front of the x (currently equal to 2) can decrease all the way down to $2 - 1 = 1$ before the solution changes, but can't increase at all. In real-life terms, this means that the amount of money you earn from tutoring can fall as low as \$1 without affecting the optimal way you should act. On the other hand, if the tutoring gig becomes any sweeter – even by 1 cent – then the optimal solution changes, in favor of more tutoring and less bartending.

Note, of course, that even though the optimal solution doesn't change, the optimal objective function value will obviously change – even though you're doing the same number of hours of tutoring in each case, the total amount you earn will change the your tutoring salary changes. But the key point is that the number of hours of each activity does not change.

The other allowable increase/decrease (for y) works in a similar way.

We're now ready to return to our problem, and to use sensitivity reports in a practical setting. In everything we will see below, the crucial first step is always to ask yourself the question

> Which constraint or objective function coefficient would change?

Once you've answered this question, you can proceed to use the sensitivity report as described above. Let's see how this applies to the questions here.

1. If the constraint that no more than 4000 bonds be purchased from company B in total were changed to 5000 bonds, it is obvious which constraint in the problem would change (the company B constraint). The change would be an increase of 1000, from 4000 to 5000. However, since all items in this questions are in thousand of dollars, the change would actually be an increase of 1, from 4 to 5.

   Now, consider that the shadow price of this constraint is -40.36. However, the required increase of 1 is significantly outside the "allowable increase" of 0.6336 for this constraint. Thus, there is no way to know *for sure* how the required amount of money would change.

   However, we can say the following:

   - The first 0.6336 of the change will cause a decrease in the amount of money we need to raise of

     $$40.36 \times 0.6336 = 25.57$$

     which corresponds to a \$25,570 decrease.

- Any further increase in the RHS of that constraint only *relaxes* the constraint further. In other words, all previous solutions are still feasible, we are only giving ourselves the *option* to buy more bonds from company B. Thus, further increase can only make our solution *better* (decrease), or stay the same.

Thus, all we can say in answer to this question is that the amount of money the trustees need to raise will decrease by *at least* $25,570.

2. It is now less obvious what part of the solution would change if the rebate were given. However, a few minutes of thought should convince you that a subsidy on Company A bonds is equivalent to a reduction in price of these bonds. Since the price of these bonds ($834.32) appears directly in the objective function, it is an objective function coefficient that would be decreasing as a result of this subsidy.

   A quick look at the sensitivity report indicates that the allowable decrease for company A year 1 bonds is $40.63. Thus, the government can subsidize these bonds by as much as $40.63 until the optimal investment plan changes.

3. Once again, our first step must be to identify the part of our model that will change as a result of this additional cash requirement. In this case, the relevant constraint is the year 7 constraint, which currently requires that the total cash flows in year 7 be greater or equal to $1,706.70 (in thousands). As a result of our additional cash requirement, this will increase by $15 (in thousands of dollars).

   Looking at the sensitivity report, the allowable increase for this constraint is $232.48. Clearly, we are within our allowable increase, and so the shadow price is valid.

   The shadow price for this constraint is 0.54. Thus, as a result of the extra capital required, the total money the trustees will need to raise in year 1 will increase by $15 \times 0.54 = 8.1$, which, converting to dollars, gives $ 8,100.[6]

[6] Note, incidentally, that this implies a yearly interest rate of

$$r = \sqrt[6]{\frac{15}{8.1}} - 1 = 10.82\%$$

which is significantly better than the 5% offered by the money-market account. This is reassuring – if it was less, it would have made more sense for the trustees to just obtain that money by investing it in the money-market account!