B6015 – Decision Models
Review Session 1

The primary aims of this review session are:

- To review the basics of optimization, as covered in class.
- To get you comfortable with formulating simple algebraic optimization problems.
- To familiarize you with the use of Excel for optimization.

In addition to the above, I hope you introduce you to a simple but powerful model for portfolio optimization. If there is time (by which I mean once everyone is absolutely and 100% satisfied that they have achieved the aims above), we will also look at some more complex situations this model is able to handle.

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One key application of decision modeling is portfolio optimization. You will be looking at a complex model for portfolio optimization in lecture 6 – in this review session we'll be warming up with a simple but nevertheless powerful model. Suppose you are an investor, with a choice of four funds to invest in. The funds are each hedged so that there is a maximum and minimum possible return in each case. The data relating to these funds are summarized in Table 1

		Today	One year from now		
	Rating	Value	Expected value	Lowest value	Best value
Fund A	AAA	50	70	59.09	82.10
Fund B	BB	5	10	0.05	36.32
Fund C	BBB	13	20	16.88	23.46
Fund D	BB	10	8	0.94	25.14

Table 1: Summary of available funds, together with ratings on your company's rating scale. The data in the table refers to a single share of each fund. To make your job simpler, expected returns have been expressed as expected fund values one year from now. All amounts are in dollars.

Your aim, of course, is to find the mix of investments that will result in a portfolio with highest expected value, while managing your downside as judiciously as you can. Assume that you have \$1,000,000 to invest, and that you would like to control risk by ensuring that your worse case outcome results in a loss of no more than \$500,000.

Assume, in addition, that due to regulation, you must invest at least $200,000 in the AAA rated fund, on the company's scale.

Part A

First, assume that no short-selling is possible, and assume that you can buy any (even fractional) number of shares of each fund. Find the portfolio that maximizes your expected outcome subject to the constraints above.

Solution

When formulating an optimization problem, you can never go wrong by following the following three simple steps[1]. Each step is, basically, a question you should ask yourself.

1. **"What am I trying to decide?"** – or, in mathematical terms, "what are my decision variables?"
2. **"What's my aim?"** – or, in mathematical terms, "what's my objective function?"
3. **"What considerations do I need to take into account?"** – or, in mathematical terms, "what are my constraints".

Let's answer these three questions in the case of this optimization problem

1. **"What am I trying to decide?"** – "what are my decision variables?"

   The first step to answering this question is trivial – you just need to read the problem. In this problem, for example, what we're trying to decide is "what to include in my portfolio".

   However, the statement "what to include in my portfolio" is not good enough, because a computer can't understand it – it's just not precise enough[2]. The crux of this first step is to reduce whatever decision you need to make to a set of unambiguous numbers – these numbers are called *decision variables*, because they fully embody the decision to be made; once you know the value the decision variables take, you know what decision will be made[3].

   In this case, how can I reduce the decision to a set of numbers? Simple – our particular decision can actually be summarized by four numbers; the number of shares of fund A, B, C and D I should include in my portfolio. So I have four decision variables, each describing the number of shares I should buy of a given fund. Once I know these four numbers, I know everything I need to know about my portfolio.

   At this point, it is worth spending some time discussing

[1] In fact, if those three steps are all you get out of this review session, I'll be happy!

[2] When I say 'not precise enough', I mean that the statement could be interpreted in a whole host of ways. Some might read it to say "what shares should I put in my portfolio", some might read it to mean "out of the four funds specified, how many shares of each should I buy", some might read it to mean "how much money should I invest and how much should I set aside", etc...In fact, only one of these interpretations is correct. We need something more precise!

[3] Note that I've slightly oversimplified the situation here. Indeed, it is sometimes necessary to include decision variables that don't necessarily correspond to a decision you have to make (some of the extra credit problems fall in this category). That said, I can't think of any situations in which this will happen in this course, and when if it does happen, it will be made so abundantly clear that you really shouldn't worry about it.

units. We could leave every number as stated in the question, but this makes the problem very unwieldy (for example, we'd have to type six zeroes every time we wanted to state the amount available to invest – \$1,000, 000). Instead, we will choose to express numbers in *thousands* – so \$1,000, 000 becomes \$1,000, and \$500,000 becomes \$500. This will make it easier to enter numbers into our model[4]. There is only one problem with this approach, however – even though expressing everything in thousands does make expressing large numbers easier, it makes fund prices much *harder* to express! Indeed, fund B, for example, now costs \$0.005 thousands of dollars. Therefore, instead of expressing fund prices in thousands of dollars, we will scale our *decision variables*. In this case, instead of our decision variables being the *number* of shares purchased, our decision variables will be the number of *thousands* of shares purchase. This will make our model completely consistent – convince yourself, as we study the rest of this problem, that the scaling does indeed work out (I have included margin notes to guide you along the way), and be sure to come and ask for help if you're having trouble understanding this.

[4]Note that casting our model into manageable units also makes it easier for Excel (or whatever engine we use) to solve the model. In the case of Risk Solver Platform, this shouldn't make a difference because the program includes a feature that automatically internally scales models to make them easier to solve. Other platforms, however, may require you to do this manually.

The last stage in step 1 is to name each decision variable, so that you can refer to them succinctly later. In this case, we will let

- x_A denote the number of thousands shares of fund A to buy.
- x_B denote the number of thousands shares of fund B to buy.
- etc...

2. **"What's my aim?"** – or, in mathematical terms, "what's my objective function?"

Once again, the first stage in answering this question is trivial. In this case, we're told our aim is to "maximize expected value of the portfolio".

Once again, however, this sentence isn't one you can directly type into Excel. Mathematically, what you need to do is express your objective (in this case expected revenue) as a function of the decision variables identified in the previous step.

In this case, this is not difficult to do. We know, for example, that each share of fund A has expected value \$70. Therefore, if we purchase x_A shares of fund A, their expected value will be $70x_A$. Using similar logic on funds B, C and D, we find that the expected value of our portfolio is $70x_A + 10x_B +$

$20x_C + 8x_D$. Thus, our objective is

$$\max \quad 70x_A + 10x_B + 20x_C + 8x_D$$

3. **"What considerations do I need to take into account?"** – or, in more mathematical terms, "what are my constraints".

   Finally, we must consider our constraints. Without constraints, this problem makes no sense – indeed, looking at the problem without constraints, nothing stops us from just making each of the decision variables extremely large, to make the objective function huge. In reality, of course, we can't do that because of budget constraints. As in the other two steps, the challenge here is to identify these constraints and then to convert them to a form Excel can understand them – ie: to convert them to something stated in terms of decision variables.

   We'll identify each of the constraints in the problem one by one, and then express them in mathematical form.

   (a) **The budget constraint** states that we have only \$1,000 (remember that we are expressing all amounts in thousands of dollars) to invest.

   How do we express this constraint mathematically? A first step is to write it as

   $$\text{Amount invested} \leq 1000$$

   Now, we need to express the LHS in terms of our decision variables. Since the current prices of funds A, B, C and D are \$50, \$5, \$13 and \$10 respectively, we can write this as[5]

   $$50x_A + 5x_B + 13x_C + 10x_D \leq 1000$$

   This is our first constraint.

   (b) **The maximum acceptable downside risk constraint** states that even in the worse case outcome, the loss incurred must be no more than \$500 (again, in thousands of dollars). We can write this as

   $$95\% \text{ worse outcome} \geq 1000 - 500$$

   How do we express the LHS in terms of our decision variables? Consider that the worse-case price of funds A, B, C and D are \$59.09, \$0.05, \$16.88 and \$0.94 respectively. As such, we can write this as

   $$59.09x_A + 0.05x_B + 16.88x_C + 0.94x_D \geq 500$$

[5] Note how the units work out – the RHS is in thousand of dollars, but each of the decision variables is expressed in thousands of shares. So $50x_A$ is the amount invested in fund 1 in *thousands* of dollars, as it must be.

(c) **The minimum AAA investment constraint** states that we must invest at least \$200 (in thousands of dollars) in the AAA fund.

This is the easiest constraint to express mathematically. The amount invested in the AAA fund (fund A) is given by $50x_A$, and so this constraint can be written

$$50x_A \geq 200$$

(d) **The fact funds cannot be shorted** implies each of our decision variables must be positive. Mathematically, we can write this as

$$x_A, x_B, x_C, x_D \geq 0$$

This concludes our enumeration of the constraints in this problem.

It is always a good idea, once a model has formulated, to summarize everything in one block. In this case, our optimization model is

$$\begin{array}{ll}
\max & 70x_A + 10x_B + 20x_C + 8x_D \\
\text{s.t.} & 50x_A + 5x_B + 13x_C + 10x_D \leq 1000 \\
& 59.09x_A + 0.05x_B + 16.88x_C + 0.94x_D \geq 500 \\
& 50x_A \geq 200 \\
& x_A, x_B, x_C, x_D \geq 0
\end{array}$$

See the Excel spreadsheet for an Excel version of this model. The optimal solution turns out to be to buy 4,000 shares of fund A, 120,318.6 shares of fund B and 15,262.09 shares of fund C, for a total expected portfolio value of \$1,788,427.

Part B

Imagine regulators now lift the AAA investment requirement. Will this improve your optimal expected portfolio value? Answer this question *without* resolving your Excel model.

Solution

At first sight, this question might seem a bit difficult – how on earth are you supposed to answer it *without* resolving your Excel model?

In fact, it's extremely easy and only requires a few seconds of thought – training yourself to think in this way will stand you in good stead for the rest of this course.

The first observation you ought to make is that removing regulatory requirement effectively *relaxes* your problem – which means it removes a constraint from your problem. This means that there are now a *larger set* of feasible solution your problem can take. Every solution that was valid in the past is still valid, but now you have even *more* valid optimal solutions. Thus, the optimal expected portfolio value can only go up – not down – because worse come to the worse, you can always just use the portfolio selected in Part A.

Now, let's take a closer look at the constraint being relaxed. Look at the optimal solution from Part A. It prescribes the purchase of 4,000 shares of fund A. Since each share of fund A costs \$50, this implies a total investment in fund A of \$200,000. This is *exactly* the limit imposed by the regulators. In other words, the constraint in question is *tight*. What this means is that it's very likely this constraint plays a part in stopping us from making more money. The fact the constraint is tight seems to imply that if we were to relax it, the amount invested in fund A would drop even further and make our profits even higher.

However, there is another possibility – and that is that the \$200,000 we invest in fund A is not a result of the constraint, but, coincidentally, *happens* to be the best possible amount invested. In that case, relaxing the constraint would make no difference.

To summarize, therefore – the fact the constraint corresponding to the regulatory requirement is tight suggests that relaxing this constraint *might* improve our portfolio, but it's not possible to say so for sure without resolving the model. However, if the constraint in question had *not* been tight, it would have been possible to say with certainty that relaxing this constraint would *not* have led to an increase in objective function.

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The remaining material will be covered only if time permits, and once everyone is comfortable with the first two parts.

Part C

Return to the situation in part A, but now assume short-selling is possible for all funds that are not AAA rated. Due to regulation, however, there are limits on the extent to which you can leverage your portfolio – the total dollar amount of shares bought must be at least four times the total dollar amount shorted. Without even touching Excel, would you expect your optimal expected portfolio price to go up or go down? Why?

Now modify your Excel model to reflect the new situation, and test out your prediction.

Solution

Once again, let us begin by applying similar logic to that in Part B. Without even touching Excel, we can tell that this modification will cause our optimal expected portfolio price to go up or stay the same. Adding the ability to short sell doesn't in any way curtail any solution that was previously feasible (so the solution in Part A is still perfectly feasible), but it adds *extra* possible portfolio. Thus, it is impossible for our portfolio to become worse – it'll either stay the same or improve.

Let's now prove this using Excel. We will, once again, go through the three steps outlined in Part A.

Decision variables . Will our decision variables change as a result of this modification? Your first instinct might be to say 'no'. Indeed, our decision is *still* completely determined by the amount we invest in each fund, except that we now allow these variables to go negative to indicate short selling.

Unfortunately, taking this approach quickly leads to difficulties[6]. Instead, we keep our current four decision variables (x_A, x_B, x_C and x_D) to denote the number of thousands of shares *purchased* and we define three *new* decision variables (which we will call y_B, y_C and y_D) to indicate the amount we short funds B, C and D (remember that the question states fund A cannot be shorted, since it is rated AAA). All variables would then still be non-negative.[7]

This is a good example of a situation in which you need some experience to properly formulate a decision model. You'll now know, forever more, that this is the best way to treat short-selling in portfolio optimization! That said, you shouldn't let this worry you – for the purposes of this course, we will never present you with a model involving such a nasty trick that you've never seen before. Any non-obvious model we give you in your final exam will be one that you've experienced in the past.

Objective function . The objective function must now be changed to include shorting. This is not particularly difficult – instead of multiplying the expected price of each fund by the corresponding x variable, we must multiply it by the *net* amount of stock in our portfolio, in this case given by $(x_B - y_B)$, for example.

Thus, our new objective is

$$\max \quad 70x_A + 10(x_B - y_B) + 20(x_C - y_C) + 8(x_D - y_D)$$

Constraints . The constraints, at least conceptually, remain as

[6] It actually works for the simplest of models, in which you just want to allow unconstrained shorting (such a model will be introduced in lecture 6, where this simple approach will be used to model short-selling). But as soon as you start including some constraints on the shorting, the method leads to more complications.

[7] People found this concept particularly tough in the review session, so it might be easier to understand it using a few examples. Imagine you're buying two shares of fund A (ie: your original variable was 2) – then you'd have $x_A = 2$ and $y_A = 0$. Imagine, instead, that you'd *shorted* three shares of fund A (ie: your original variable was -3) – then you'd have $x_A = 0$ and $y_A = 3$. In either case, you can always recover the *net* amount invested by calculating the quantity $x_A - y_a$.

before. However, their implementation will change. Let's have a look at each constraint one by one

The budget constraint remains the same, but it must now reflect the *net* stock position rather than just the long position (this is very similar to the change we carried out with the objective function). Thus, replacing the x decision variables with the *net* position, the constraint becomes

$$50x_A + 5(x_B - y_B) + 13(x_C - y_C) + 10(x_D - y_D) \leq 1000$$

The regulatory constraint is the only one that's not affect, since it's impossible to short stock A. It remains as it was before

$$50x_A \geq 200$$

The worse downside constraint *will* change, and unfortunately, the change isn't as simple as replacing the amount longed by the net stock bought. The reason it's not that simple is because 'worse downside' means different things for bought stocks and shorted stocks.

When you *buy* a stock, you hope that it'll go *up*. Therefore, your worst-case scenario is the *lowest* price that the stock might take in the future. On the other hand, if you *short* a stock (ie: if you sell it with the intention of buying it back in the future), you're hoping it'll go *down*, so that you can buy it later at a lower price. Thus, for a shorted stock, the 'worst-case scenario' is the *highest* price that the stock might take in the future.

Taking these facts into consideration, we end up with a constraint that looks something like this

$$59.09x_A + 0.05x_B + 16.88x_C + 0.94x_D - 36.32y_B - 23.46y_C - 25.14y_D \geq 500$$

The first part of the constraint is identical to what we had previously. The second part takes into account the worse-case value of *shorted* stocks.

In addition to the above, there will be one more constraint to reflect the fact that the amount longed must be at least four times as large as the amount shorted, at the time of portfolio purchase. This is easy to express as

$$50x_A + 5x_B + 13x_C + 10x_D \geq 4 \times (5y_B + 13y_C + 10y_D)$$

Thus, our new optimization problem is

$$
\begin{array}{ll}
\max & 70x_A + 10(x_B - y_B) + 20(x_C - y_C) + 8(x_D - y_D) \\
\text{s.t.} & 50x_A + 5(x_B - y_B) + 13(x_C - y_C) + 10(x_D - y_D) \leq 1000 \\
& 59.09x_A + 0.05x_B + 16.88x_C + 0.94x_D - 36.32y_B - 23.46y_C - 25.14y_D \geq 500 \\
& 50x_A \geq 200 \\
& 50x_A + 5x_B + 13x_C + 10x_D \geq 4 \times (5y_B + 13y_C + 10y_D) \\
& x_A, x_B, x_C, x_D, y_B, y_C, y_D \geq 0
\end{array}
$$

See the Excel spreadsheet for an Excel version of this model. The optimal portfolio buys 4,000 shares of fund A, 57,425.03 shares of fund B, 65,092.94 shares of fund C and shorts 33,333.33 shares of fund D. The optimal expected portfolio price is \$1,889,442, rather better than what we were able to obtain in Part A.

Part D

Returning to the model of Part A (no short selling) suppose that regulators have agreed to relax the requirement to invest \$500,000 in the AAA rated funds, provided you take out adequate insurance to cover losses. Your analysts estimate that each dollar under \$200,000 you choose not to invest in a AAA rated fund will cost you 8 cents in insurance, payable at the time of purchase.

Incorporate this into your model, and find the new optimal portfolio.

Solution

Once again, let's go through the three optimization steps.

Decision variables . We once again return to the situation in Part A, with 4 decision variables. Consider, however, that are solution is now no longer totally determined by these 4 decision variables. It's no longer enough to know how much we are investing in each fund – we also need to know how much insurance we'll be purchasing (or, equivalently, by how many dollars we'll be relaxing our regulatory requirement).

This requires a new decision variable, and we call it I (for 'insurance'). It denotes the number of *thousands* of units of insurance we decided to buy (ie: the number of thousands of dollars we decided to shave off our regulatory requirement).

Objective function . This stays the same. Our objective is still to maximize the expected worth of our portfolio, which doesn't directly involved insurance.

If you approached this problem like I did, you might have been tempted to simply subtract the price of the insurance from the objective function. However, this is *not correct*. Why? Because the question clearly specifies that the insurance must be paid for at the time of purchase. This means that buying insurance doesn't only take away from your profits, it also means you have less money available to invest. Furthermore, deducting the insurance price from your expected portfolio value doesn't take the time-value of money into account.

Constraints . The constraints are all as in Part A, with two exceptions. Obviously, the regulatory constraint must be modified to reflect any insurance purchased. The budget constraint must also be modified to reflect the fact that any insurance bought reduces the amount we can invest.

Modifying our regulatory constraint is easy – it simply becomes

$$50x_A \geq 200 - I$$

The RHS now reflects the *new* constraint.

The budget constraint is also not too hard. The amount spent on insurance is equal to $0.08I$ thousands of dollars, and so the total amount left to invest is $1000 - 0.08I$. As such, our budget constraint becomes

$$50x_A + 5x_B + 13x_C + 10x_D \leq 1000 - 0.08I$$

Thus, our new optimization problem is

$$\begin{array}{ll} \max & 70x_A + 10x_B + 20x_C + 8x_D \\ \text{s.t.} & 50x_A + 5x_B + 13x_C + 10x_D \leq 1000 - 0.08I \\ & 59.09x_A + 0.05x_B + 16.88x_C + 0.94x_D \geq 500 \\ & 50x_A \geq 200 - I \\ & x_A, x_B, x_C, x_D \geq 0 \end{array}$$

The optimal solution is to buy the maximum amount of insurance ($200,000 units), 120,715.5 shares of fund B and 29,263.28 shares of fund C, for a total expected portfolio value of $1,792,420.