Green's Functions

Generalised Functions

- The **Heaviside Step Function** is defined as

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

  (We can arbitrarily define the value of *H(0)* to be ½).
- Consider the following "top-hat" function

$$\delta_\varepsilon(x) = \begin{cases} 1/\varepsilon & 0 < x < \varepsilon \\ 0 & otherwise \end{cases}$$

  This can be written as

$$\delta_\varepsilon(x) = \frac{H(x) - H(x-\varepsilon)}{\varepsilon}$$

- The total area under the top hat curve is always equal to 1, and we have that:

$$\int_{-\infty}^{x} \delta_\varepsilon(\xi)\,\mathrm{d}\xi = \begin{cases} 0 & x \le 0 \\ x/\varepsilon & 0 \le x \le \varepsilon \\ 1 & x \ge \varepsilon \end{cases}$$

- As $\varepsilon \to 0$, the function becomes an infinite spike at the origin, but still with area 1 underneath it, and its integral becomes

$$\int_{-\infty}^{x} \delta_\varepsilon(\xi)\,\mathrm{d}\xi = H(x)$$

  In other words

$$\delta(x) = H'(x)$$

- This is very useful when physically representing something that is localised at a point and whose "overall sum under the curve" is known – impulse, for example.
- Another way to define the delta function is as one that satisfies the following property

$$\int_{-\infty}^{\infty} f(x)\delta(x-\xi)\,\mathrm{d}x = f(\xi)$$

  (Where the integral can be taken over any interval that includes $x = \xi$).

- We can use this generalised expression to prove the fact that $\delta(x) = H'(x)$, by simply integrating $f(t)H'(t)$ between $\pm\infty$.
- The Dirac delta function can also be viewed as the limit of localised functions other than the top-hat used above. In such cases, the function might not "look" like a spike, but it will nevertheless behave like the Dirac delta. For example,
$$\lim_{\varepsilon \to 0}\left[\frac{1}{\pi x}\sin\left(\frac{x}{\varepsilon}\right)\right] = \delta(x)$$
But the central peak still stays broad. This can be understood in terms of the fact that off the origin, the "wiggles" cancel each other out to 0, and only at the peak do we keep anything when "picking out a value". [See also notes in the Fourier Transform section].
- The derivative of the Delta function has both positive and negative "spikes" at $x = 0$. The defining property of $\delta'(x)$ can be taken to be
$$\int_{-\infty}^{\infty} f(x)\delta'(x-\xi)\,\mathrm{d}x = -\int_{-\infty}^{\infty} f'(x)\delta(x-\xi)\,\mathrm{d}x = -f'(\xi)$$
(This can easily be proven using integration by parts). What the two spikes are effectively doing is picking the "difference" between the function at two very close points…
- In general, **integrating** the δ function or one of its integrals makes it *smoother*. **Differentiating** it increases the discontinuities. For example – $\int \delta$ is discontinuous itself. $\int\int \delta$ is continuous but with a discontinuous *first* derivative. $\int\int\int \delta$ is continouous, but with a discontinuous *second* derivative, etc…
- Two generalised functions of the **same** variable **cannot** be **multiplied** together.
- When proving results about generalised functions, the following tips help:

  - The key is to find an integral based on the generalised definition of the function that evaluates to something we want for ***any*** function *f*.
  - It is important to remember that the variable of integration <u>must</u> be the argument of the generalised function – otherwise, it doesn't work.

Differential Equations Containing δ

- If a differential equation involves a step or delta function, this implies a lack of smoothness in the solution.
- The equation must be solved on either side of the discontinuity (ie: producing four arbitrary constants) and the two parts connected with appropriate matching conditions (ie: getting rid of two of these constants).
- <u>*Note*</u>: if boundary or initial conditions are given, the process can be simplified by writing down solutions on either side of the discontinuity that satisfy their respective boundary condition.
- To find these "matching conditions", we note that the **highest order derivative** on the LHS must be of the same "order of discontinuity" as the RHS. This is because differentiating a discontinuous function makes the discontinuity *more severe* – therefore:
  - If the highest order derivative on the LHS is *more continuous* than the RHS, then so will all the lower-order derivatives, and the equality won't be satisfied.
  - If the highest order derivative on the LHS *less continuous* than the RHS, then the equality definitely won't be satisfied, because none of

the lower terms will "fix" that high degree of discontinuity.

- We can therefore determine the *class* of function the solution falls into.
- We can then determine **jump conditions** by integrating both sides of the differential equation from ε before the discontinuity to ε after, and letting $\varepsilon \to 0$. By definition, only the second highest derivative in the integral will contribute to the jump. We denote:
$$[f(x)] \equiv \lim_{\varepsilon \to 0} [f(x)]_{x=\xi-\varepsilon}^{x=\xi+\varepsilon}$$
$[f(x)]$ is, effectively, the "jump" in the function at $x = \xi$.

Definition of Green's Function

- Consider the equation
$$\frac{\partial^2 y}{\partial x^2} + p\frac{\partial y}{\partial x} + qy = f(x) \qquad\qquad a \le x \le b$$
And define the linear operator $\wp$ such that we can write this as
$$\wp(y) = f(x) \qquad\qquad a \le x \le b$$
- Now, let there be a function G (the **Green's Function**) such that the general solution of this equation is
$$y(x) = \int_a^b G(x,\xi) f(\xi)\ \mathrm{d}\xi \qquad\qquad (*)$$
If we apply the differential operator to both sides, we end up with:
$$\wp[y(x)] = \int_a^b \left(\wp[G(x,\xi)]\right) f(\xi)\ \mathrm{d}\xi$$
$$f(x) = \int_a^b \left(\wp[G(x,\xi)]\right) f(\xi)\ \mathrm{d}\xi$$
This, however, is precisely the property of the Dirac delta function. As such, we require that
$$\wp[G(x,\xi)] = \delta(x-\xi)$$
G is, effectively, the **response** of the system to a **unit impulse** at $x = \xi$ (since when the linear operator is applied, it gives a peak at that point). The solution

then consists of these responses **weighed** by the **actual impulse** at each point (given by *f(x)* at that point).

- G must satisfy two other conditions:
  - It must be defined such that the solution y produced by (*) satisfies the boundary conditions. If these conditions are **homogenous**, inspection of (*) shows that we simply need to make G also satisfy these boundary conditions.
  - The second condition involves the discontinuities of the function at $x = \xi$. Using the argument developed above, the function must be continuous but have a discontinuity in its first derivative. This means that at $x = \xi$

$$[G] = 0 \qquad \left[\frac{\partial G}{\partial x}\right] = 1$$

- We then simply find G by solving the differential equation on both sides of the discontinuity and applying the right matching conditions (using matrices is often helpful).
- A matrix-approach to finding Green's Function leads to the following results:
  - For a boundary-value problem

$$G(x,\xi) = \begin{cases} \frac{y_a(x) y_b(\xi)}{W(\xi)} & a \le x \le \xi \\ \frac{y_a(\xi) y_b(x)}{W(\xi)} & \xi \le x \le b \end{cases}$$

    Where:
    - a and b are the positions of the boundaries.
    - y_a and y_b are complementary functions of L satisfying the boundaries at a and b respectively.
    - W is the **Wronskian**, given by

$$W[y_a, y_b](x) = \begin{vmatrix} y_a & y_b \\ y_a' & y_b' \end{vmatrix}$$

  - For an initial-value problem

$$G(x,\xi) = \begin{cases} 0 & a \le x \le \xi \\ \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{W(\xi)} & \xi \le x \le b \end{cases}$$

Where:

- a and b are, again, the boundaries of the problem.
- y_1 and y_2 are two **linearly independent** complimentary functions of the equation.
- W is as above.

Miscellaneous

- Green's Function can also be used to find the **general** solution of a differential equation as follows:
  - First, find a solution to the associated **complementary equation**, which will include two constants.
  - Then, find a **particular integral** by taking arbitrary homogenous boundary conditions (eg: $y(0) = y'(0) = 0$).

