Fourier Transforms

Introduction

- A **periodic signal** can be analysed into its harmonic components by calculating its **Fourier Series**. If the period is P, then the harmonics have frequency n/P, where n is an integer.
- The **Fourier Transform** generalises this idea to functions that are not periodic; the 'harmonics' can have *any* frequency.

Fourier Series

- A function *f(x)* with period P (ie: *f(x + P) = f(x)* for all x) can be written as a **Fourier Series**

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n x) \qquad (*)$$

Where $k_n = 2\pi n / P$ is the **wavenumber** of the **n^{th} harmonic** and

$$a_n = \frac{2}{P}\int_{\text{over one period}} f(x)\cos(k_n x)\ \mathrm{d}x$$
$$b_n = \frac{2}{P}\int_{\text{over one period}} f(x)\sin(k_n x)\ \mathrm{d}x$$

- This can be proved by multiplying both sides of (*) by $\cos(k_n x)$ and $\sin(k_n x)$, integrating over a period and using the orthogonality relations:

$$\int_{\text{one period}} \sin(k_r x)\cos(k_p x)\ \mathrm{d}x = 0 \qquad \forall\ r, p$$

$$\int_{\text{one period}} \sin(k_r x)\sin(k_p x)\ \mathrm{d}x = \begin{cases} \frac{1}{2}L & r = p > 0 \\ 0 & \textit{otherwise} \end{cases}$$

$$\int_{\text{one period}} \cos(k_r x)\cos(k_p x)\ \mathrm{d}x = \begin{cases} L & r = p = 0 \\ \frac{1}{2}L & r = p > 0 \\ 0 & r \neq p \end{cases}$$

- It saves time to remember that
  - For **even** functions, all the $\boldsymbol{b_n = 0}$.

  - For **odd** functions, all the $\boldsymbol{a_n = 0}$.
- Fourier Series can also be found by differentiation and integration:
  - Integrating or differentiating a Fourier series term-by-term leads to the differential or the integral of the original function.
  - It is important to remember the arbitrary constant when integrating.
  - When integrating, the result will include an 'x' term. This means that the expression isn't, strictly speaking, a Fourier series. An alternative expression needs to be found for the x term (perhaps by differentiating…)
- If we define

$$c_n = \frac{1}{2}\begin{cases} a_{-n} + ib_{-n} & n < 0 \\ a_0 & n = 0 \\ a_n - ib_n & n > 0 \end{cases}$$

Then we can express our results more simply in terms of a **complex Fourier series**:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \qquad (\#)$$

Where k_n is as above, and

$$c_n = \frac{1}{P}\int_{\text{one period}} f(x)e^{-ik_n x}\ \mathrm{d}x$$

If $f(x)$ is real, then $c_{-n} = c_n^*$.
- This can be proved by multiplying both sides of (#) by $e^{-ik_n x}$, integrating over a period, and using the following orthogonality relation:

$$\frac{1}{P}\int_{\text{one period}} e^{i(k_n - k_m)x}\,\mathrm{d}x = \delta_{mn}$$

Fourier Transforms

- Consider the complex forms of Fourier series, which can be written as follows:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{P} \int_{\text{one period}} f(u) e^{-ik_n u} \, \mathrm{d}u \; e^{ik_n x}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} \int_{\text{one period}} f(u) e^{-ik_n u} \, \mathrm{d}u \; e^{ik_n x}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} g(k_n) \; e^{ik_n x}$$

Where $\Delta k = 2\pi / P$ is the difference between fundamental wavenumbers.

- Now, as we transit from a periodic function to a non-periodic function ($P \to \infty$), $\Delta k \to 0$. Then, by the definition of the integral, the above becomes

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k) e^{ikx} \, \mathrm{d}k$$

- Substituting *g(k)* back in, we get **Fourier's Inversion Theorem**:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u) e^{-iku} \, \mathrm{d}u \right] e^{ikx} \, \mathrm{d}k$$

(Note how the limits of the inner integral become ∞ as a result of the fact $T \to \infty$).

- Using this result, we can define the **Forward Fourier Transform** as

$$\boxed{\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, \mathrm{d}x}$$

And the **Inverse Fourier Transform** as

$$\boxed{f(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, \mathrm{d}k}$$

- For $\tilde{f}(k)$ to exist, it is sufficient that $f(x)$
  - Have **bounded variation**
  - Have a **finite** number of **discontinuities**
  - Be **absolutely integrable**

  $$\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x < \infty$$

  (This implies that $f(x) \to 0$ as $x \to \pm\infty$).

Fourier Transforms & δ-functions

- We can write Fourier's Inversion Theorem as

$$f(x) = \int_{-\infty}^{\infty} f(u) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-u)} \, \mathrm{d}k \right] \mathrm{d}u$$

Comparison with the defining property of the Dirac delta reveals that

$$\delta(x-u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-u)} \, \mathrm{d}k$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm ikx} \, \mathrm{d}k$$

(This can also easily be verified using the substitution property of the Dirac delta). Points:

- This definition is not mathematically rigorous, because the integral diverges. However, by Fourier's Inversion Theorem, the integral behaves as the Dirac delta should, and so the equality can be considered to hold.
- This suggests that the Dirac delta can also be represented as the superposition of a complete spectrum of harmonic waves – in other words, as the transform of the function *f(x) = 1*. This can be used to find the Dirac delta as the limiting function of a number of other distributions.
- The $\pm$ in the expression can be obtained by writing Fourier's inversion theorem differently, and confirms that:
    - $\delta(x) = \delta(-x)$
    - $\delta(x) = \delta^*(x)$, which means that the Dirac delta is real.

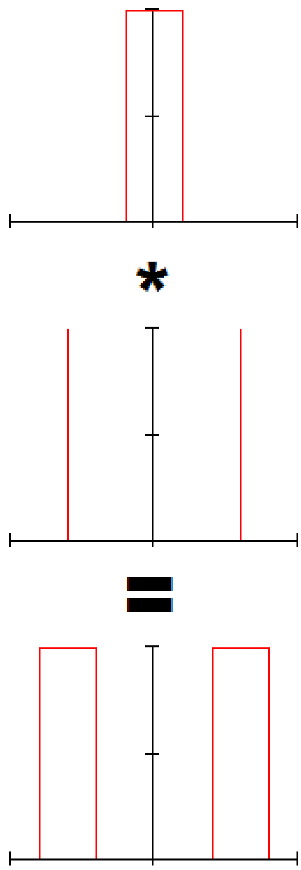
Properties of Fourier Transforms

- **Linearity**
    - $\widetilde{\alpha f(t)} = \alpha \tilde{f}(\omega)$
    - $\widetilde{f(t) + g(t)} = \tilde{f}(\omega) + \tilde{g}(\omega)$
- **Scaling** (real α) – $\widetilde{f(\alpha t)} = \frac{1}{|\alpha|} \tilde{f}\left(\frac{\omega}{\alpha}\right)$
- **Shift/Exponential** (real α)

- $\widetilde{f(t-\alpha)} = e^{-ik\alpha}\tilde{f}(\omega)$
- $\widetilde{e^{i\alpha x} f(t)} = \tilde{f}(k-\alpha)$

- **Differentiation/Multiplication**
  - $\widetilde{f'(t)} = i\omega\tilde{f}(\omega)$
  - $\widetilde{tf(t)} = i\tilde{f}'(k)$
- **Integration** – $\widetilde{\int^t f(s)\,\mathrm{d}s} = \frac{1}{i\omega}\tilde{f}(\omega) + 2\pi c\delta(\omega)$ [last term is for the arbitrary constant].
- **Duality** – $\widetilde{\tilde{f}(t)} = 2\pi f(-\omega)$
- **Complex conjugation and parity inversion** – $\left[\widetilde{f(t)}\right]^* = \left[\tilde{f}(-\omega)\right]^*$
- **Symmetry** – $f(-t) = \pm f(t) \Leftrightarrow \tilde{f}(-\omega) = \pm\tilde{f}(\omega)$

Convolution



- The **convolution** of two functions, $h = f * g$, is defined by

$$h = [f * g](x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,\mathrm{d}y$$

  (The **sum** of the **arguments** of f and g is the argument of h).
- **Convolution** is a **symmetric operation**, and is also **associative** and **distributive**.
- Convolution of a function with a Dirac delta can be viewed as "copying over" the function at the position of each delta function (see margin).
- Convolution can also be viewed as the result of uncertainty introduced by an apparatus…
  - Let the actual function we're attempting to measure has probably distribution *f(x)* (ie: probably true values lies between x and $x + \delta x$ is $f(x)\delta x$).
  - Let the apparatus have a resolution function *g(y)* (ie: probably that a value y will in fact be observed elsewhere in the range $y + \delta y$ is $g(y)\delta y$).

- Then the actual observed distribution is $h(z) = [f * g](z)$.

- The **Convolution Theorem**, concerning a function $h(x) = [f * g](x)$ can be proved as follows:

$$\begin{aligned}
\tilde{h}(\omega) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y)g(x-y)\,\mathrm{d}y\right]e^{-i\omega x}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)g(x-y)e^{-i\omega x}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)g(z)e^{-i\omega y}e^{-i\omega z}\,\mathrm{d}z\,\mathrm{d}y \\
&= \left[\int_{-\infty}^{\infty} f(y)e^{-i\omega y}\,\mathrm{d}y\right]\left[\int_{-\infty}^{\infty} g(z)e^{-i\omega z}\,\mathrm{d}z\right] \\
&= \tilde{f}(\omega)\tilde{g}(\omega)
\end{aligned}$$

In other words, **convolution in real space** is equivalent to **multiplication in Fourier space**. In summary:

$$\widetilde{[f * g](x)} = \tilde{f}(x)\tilde{g}(x)$$

$$\widetilde{f(x)g(x)} = \tfrac{1}{2\pi}\left[\tilde{f} * \tilde{g}\right](x)$$

- This also implies that **deconvolution** can be achieved by **division** in the Fourier domain, using the first form of the Theorem.

Correlation

- The **correlation** of two functions, $h = f \otimes g$, is given by

$$h = [f \otimes g](x) = \int_{-\infty}^{\infty} [f(y)]^* \, g(x+y)\,\mathrm{d}y$$

(The **difference** of the arguments of f and g is now the argument of h).

- The **correlation** is a way of quantifying the **relationship between two functions** as one is **displaced with respect to the other** by a distance ***z***. If the functions are **in-phase** with each other, the correlation is **positive** and vice-versa.

- Correlation is **associative** and **distributive**, but **<u>not</u> commutative**. In fact:

$$[f \otimes g](z) = [g \otimes f]^*(-z)$$

- If $h = f \otimes g$, then
$$\tilde{h}(\omega) = \left[\tilde{f}(\omega)\right]^* \tilde{g}(\omega)$$
This is the **Wiener-Khinchin Theorem**.
- A corollary of the Wiener-Khinchin Theorem concerns the Fourier Transform of the **autocorrelation function**:
$$\widetilde{[f \otimes f]}(\omega) = \left|\tilde{f}(\omega)\right|^2 = \Phi(\omega)$$
This quantity is known as the **power spectrum** of *f*. This is often used to quantify the spectral content of a signal *f(t)*.
    - For a **perfectly periodic signal**, the power spectrum consists of series of delta functions at the harmonics.
    - **White noise** is an ideal random signal with autocorrelation function proportional to $\delta(t)$ - it is **perfectly decorrelated** (unless they're right on top of each other, at 0). As such, the spectrum is simply flat.

Parseval's Theorem

- If we apply the inverse transform to the Wiener-Khinchin Theorem, we obtain:
$$\int_{-\infty}^{\infty} [f(y)]^* g(x+y)\ \mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\tilde{f}(\omega)]^* \tilde{g}(\omega) e^{i\omega x}\ \mathrm{d}\omega$$
Letting $x = 0$, and re-labelling $y \to x$:
$$\int_{-\infty}^{\infty} [f(x)]^* g(x)\ \mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\tilde{f}(\omega)]^* \tilde{g}(\omega)\ \mathrm{d}\omega$$
If $g = f$:
$$\int_{-\infty}^{\infty} \left|f(x)\right|^2\ \mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\tilde{f}(\omega)\right|^2\ \mathrm{d}\omega$$