Dynamics – Rigid Body Dynamics

Introduction

- A **rigid body** is a **many-particle system** in which the **distance between particles** is **fixed**. The **location** of **all** particles is described by **6 coordinates** – **3 spatial** and **3 angular**.
- The **velocity** is determined by $\boldsymbol{v}$, the velocity of the **CoM** and $\boldsymbol{\omega}$, the **angular velocity**.
- The basic two equations of angular motion are

$$\boxed{M\ddot{\boldsymbol{R}} = \boldsymbol{F}_0}$$

  The **centre of mass** moves as if it were a **single particle** under the action of a force $\boldsymbol{F}_0$.

$$\boxed{\dot{\boldsymbol{J}} = \boldsymbol{G}_0}$$

  The **rate of change of angular momentum** is equal to the **total applied couple**.
- Other basic equations:
  - The velocity $\boldsymbol{v}$ of a particle at a distance $\boldsymbol{r}$ from an axis around which a rotation at speed $\boldsymbol{\omega}$ is happening is

$$\boxed{\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}}$$

  - For similar reasons:

$$\boxed{\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \boldsymbol{\omega} \times \boldsymbol{J}}$$

  - Angular speeds are **additive**. To if *frame 1* is rotating with $\boldsymbol{\omega}_{1 \text{ wrt } 2}$ with respect to *frame 2*, which is rotating with $\boldsymbol{\omega}_{2 \text{ wrt } 3}$ with respect o *frame 3*, then

$$\boxed{\boldsymbol{\omega}_{1 \text{ wrt } 3} = \boldsymbol{\omega}_{1 \text{ wrt } 2} + \boldsymbol{\omega}_{2 \text{ wrt } 3}}$$

Relating $\boldsymbol{J}$ and $\boldsymbol{\omega}$

- If the body is rotating at $\boldsymbol{\omega}$, the **total angular momentum** is given by

$$\begin{aligned} \boldsymbol{J} &= \sum \boldsymbol{r} \times \boldsymbol{p} \\ &= \sum \boldsymbol{r} \times m(\boldsymbol{\omega} \times \boldsymbol{r}) \\ &= \sum m\left[r^2\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{r})\boldsymbol{r}\right] \\ &= \sum m\left[r^2\boldsymbol{\omega} - (\omega_x x + \omega_y y + \omega_z z)\boldsymbol{r}\right] \end{aligned}$$

In detail

$$\boldsymbol{J} = \underbrace{\begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{pmatrix}}_{\boldsymbol{I}} \boldsymbol{\omega}$$

$$\boxed{\boldsymbol{J} = \boldsymbol{I}\boldsymbol{\omega}}$$

[*The non-diagonal elements are fairly easy to derive. The diagonal ones should actually have $x^2 + y^2 + z^2$, because one of the terms is always knocked out by the second term in the sum*]. In other words, $\boldsymbol{J}$ is **proportional** to $\boldsymbol{\omega}$, but **not necessarily parallel to it**.

- The off-axes elements are rather hard to understand – they correspond to the fact that looking at a particle at a given instant, it's impossible to tell exactly around which axis it's moving.
- Also, we can find the **kinetic energy**

$$\begin{aligned} T &= \sum \tfrac{1}{2} m\left[(\boldsymbol{\omega}\times\boldsymbol{r})\cdot(\boldsymbol{\omega}\times\boldsymbol{r})\right] \\ &= \sum \tfrac{1}{2} m\left[\boldsymbol{\omega}\cdot\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})\right] \end{aligned}$$

$$\boxed{T = \tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{J}}$$

- The couple is then given by

$$\boxed{\boldsymbol{G} = \dot{\boldsymbol{J}} = \boldsymbol{\omega}\times\boldsymbol{J}}$$

- Note that $\boldsymbol{I}$ must be specified **with its origin** and **with its set of axes**.

Properties of $\boldsymbol{I}$

- $\boldsymbol{I}$ is a **symmetric tensor**. It therefore has **three real eigenvalues** and **three perpendicular eigenvectors**.
- With respect to the **eigenvector basis**:

$$\boldsymbol{I}' = \begin{pmatrix} I_1 & \cdot & \cdot \\ \cdot & I_2 & \cdot \\ \cdot & \cdot & I_3 \end{pmatrix}$$

$$\boldsymbol{J}_\alpha = I_\alpha \omega_\alpha \qquad \text{[No sum]}$$

$$T = \tfrac{1}{2} I_\alpha \omega_\alpha^2 \qquad \text{[Sum]}$$

- The **eigenvector axes** are called the **principal axes**, and the *I*s are called the **principal moments of inertia**.
- An **alternative** way to think of this is that the **principal axes** are ones around which objects are "happy" to rotate **without any torque being applied**.
- In ω-space, surfaces of constant T form an **ellipsoid**, with **axes** of length $\propto I_\alpha^{-1/2}$. Also, in ω-space:

$$\text{grad } T = I_\alpha \omega_\alpha = \boldsymbol{J}$$

  So $\boldsymbol{J}$ is **perpendicular** to **surfaces of constant** $\boldsymbol{T}$ at $\boldsymbol{\omega}$.
- We can classify the **principal axes** as follows:
  - **Spherical tops** – all the I are equal, and $\boldsymbol{J} = I\boldsymbol{\omega}$, with I **scalar**. The body is **isotropic** with the **same** I about **any axis** (eg: *sphere, cube*).
  - **Symmetrical tops** – $I_1 = I_2 \neq I_3$. $\mathbf{e_3}$ axis is **unique**, but $\mathbf{e_1}$ and $\mathbf{e_2}$ are any two **mutually perpendicular** vectors **perpendicular** to $\mathbf{e_3}$ (eg: *lens, cigar*).
  - **Asymmetrical tops** – all *I*s different, and axes are unique.
- Consider **any two *I*s**:

$$I_1 + I_2 = \sum m(y^2 + z^2 + x^2 + z^2) = I_3 + 2\sum mz^2 \geq I_3$$

  So no I can be larger than the sum of the other two. Furthermore, if $\boldsymbol{z = 0}$ for **every particle** (ie: if we have a **lamina**), then

$$I_3 = I_1 + I_2$$

- Consider an axis at a distance **a** away from a **principal axis** and **parallel to it**, and let **r** be the distance of each particle from the **principal axis**. Then:

$$I = \sum m(\boldsymbol{r} + \boldsymbol{a}) \cdot (\boldsymbol{r} + \boldsymbol{a}) = I_0 + Ma^2 + 2 \underbrace{\left(\sum m\boldsymbol{r}\right)}_{\substack{=0 \text{ when } \boldsymbol{r} \text{ measured} \\ \text{relative to C of M}}} \cdot \boldsymbol{a} = I_0 + Ma^2$$

This is the **Parallel Axis Theorem**, where each vector is considered to be a **projection** in a plane **perpendicular** to the **axes**.

Two Basic Problems

- You **whack it** – what happens? Steps for solution:
  - Define principal axes with a sensible origin.
  - Calculate an expression for $\boldsymbol{J}$ in terms of the impulse:
  $$\boldsymbol{J} = \int \boldsymbol{\tau}\,\mathrm{d}t = \int \boldsymbol{r}_B \times \boldsymbol{F}\,\mathrm{d}t = \boldsymbol{r}_B \times \int \boldsymbol{F}\,\mathrm{d}t = \boldsymbol{r}_B \times \boldsymbol{P}$$
  Where B is the point at which the whack occurred, and $\boldsymbol{r_B}$ can be taken out of the integral because the whack is assumed to be instantaneous.
  - Work out an expression for $\boldsymbol{J}$ in terms of $\boldsymbol{\omega}$, using the moments of inertia.
  - Equate the two expressions for $\boldsymbol{J}$.
  - Work out the motion of the CM using standard linear mechanics.
  - **NOTE**: The obvious origin to use is the CM, but other origins *can* be used subject to the provisos above for using $\boldsymbol{\tau} = \dot{\boldsymbol{J}}$. So a pivot, for example, is fine to use.
- You **apply a torque** – what's the **frequency of rotation**?
  - Define principal axes with a sensible origin (eg: the CM – see above).
  - Find an expression for $\boldsymbol{\omega}$ in these axes (with unknown magnitude), and find a corresponding expression for $\boldsymbol{J}$, using the principal moments of inertia.
  - Find $\mathrm{d}\boldsymbol{L}/\mathrm{d}t = \boldsymbol{\omega} \times \boldsymbol{L}$.
  - Calculate the **torque** ($= \boldsymbol{r} \times \boldsymbol{F}$) and equate it with $\mathrm{d}\boldsymbol{L}/\mathrm{d}t$.

Free Motion – Euler's Equation

- **Free precession** is a situation in which $\boldsymbol{F} = \boldsymbol{0}$ and $\boldsymbol{G} = \boldsymbol{0}$. In such a case, $\boldsymbol{J}$ is constant. $\boldsymbol{\omega}$ is constant if $\boldsymbol{J}$ is along one of the **principal axes**, but otherwise, it will change **direction**, and perhaps even **magnitude**.
- We use the **Euler Equations** to analyse this problem.

- The rate of change of angular momentum vector in the principal-axes frame (which is rotating around with the body) and the lab frame are related by
$$\left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_{\text{lab}} = \left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_{\text{PA}} + \boldsymbol{\omega} \times \boldsymbol{J}$$
- Now, let's assume that a couple $\boldsymbol{G}$ is being applied in the **lab** frame. We know that
$$\boldsymbol{G} = \left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_{\text{lab}}$$
Therefore, using the equations above:
$$\boldsymbol{G} = \left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_{\text{PA}} + \boldsymbol{\omega} \times \boldsymbol{J}$$
- Finally, we note that in the principal axes frame, $\boldsymbol{J} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$. Therefore, casting **both sides** of this equation into the principal axes frame only
$$\boxed{\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2}$$
And similarly with any **cyclic permutation** of indices.
- A few notes:
  - **All** the **quantities** in this equation are **measured** with respect to the **body frame** (which is moving). This is the **advantage** of these equations – all we have to consider is the forces that the body "**feels**".
  - The **two terms** of the **RHS** refer to two types of ways $\boldsymbol{J}$ can change – because it can change **in the body frame** and also because the **body frame** is **itself rotating**.

Free Motion – Examples

- **FREE SYMMETRIC TOP**
  - For a **symmetrical top** ($I_1 = I_2 = I$) which is **free in space** (ie: **no torque**) the **Euler Equations** become
$$I\dot{\omega}_1 + (I_3 - I)\omega_3\omega_2 = 0$$
$$I\dot{\omega}_2 + (I - I_3)\omega_1\omega_3 = 0$$
$$I_3\dot{\omega}_3 = 0$$
  - The last equation implies ω_3 is **constant**. Let's define

$$\Omega = \frac{I_3 - I}{I}\omega_3$$

Then the general solution of the first two equations becomes:

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = A\begin{pmatrix}\cos[\Omega t + \phi]\\ \cos[\Omega t + \phi]\end{pmatrix}$$

- **Interpretation from the body frame**
  - In the body frame, ω_1 and ω_2 seem to form a **circle** in the x–y plane, with **frequency** Ω. How **high** that circle is depends on $\boldsymbol{\omega}_3$.
  - $\boldsymbol{L}$ could be **above** $\boldsymbol{\omega}$ (if $I_3 > I$ – an **oblate top**) or **below** $\boldsymbol{\omega}$ (if $I_3 < I$ – a **prolate top**).
- **Interpretation from the fixed lab frame**
  - In that case, the Euler Equations are useless, because they deal with the body frame, so we express things from scratch, but **in terms of the body frames**:

$$\boldsymbol{\omega} = \left(\omega_1\hat{\boldsymbol{x}}_1 + \omega_2\hat{\boldsymbol{x}}_2\right) + \omega_3\hat{\boldsymbol{x}}_3$$

$$\boldsymbol{L} = I\left(\omega_1\hat{\boldsymbol{x}}_1 + \omega_2\hat{\boldsymbol{x}}_2\right) + I_3\omega_3\hat{\boldsymbol{x}}_3$$

$$\Downarrow$$

$$\boxed{\boldsymbol{\omega} = \frac{\boldsymbol{L}}{I} - \Omega\hat{\boldsymbol{x}}_3}$$

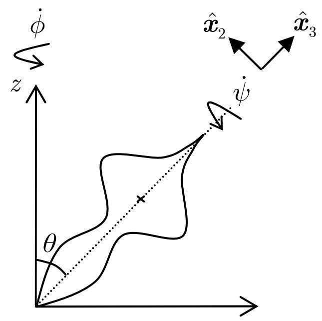
    With Ω defined as above.
  - This **linear relationship** between $\boldsymbol{\omega}$, $\boldsymbol{L}$ and $\hat{\boldsymbol{x}}_3$ implies that they are in the **same plane**.
  - Furthermore, the **rate of change of** $\hat{\boldsymbol{x}}_3$ is $\boldsymbol{\omega}\times\hat{\boldsymbol{x}}_3$, because it **only** changes as a result of the rotation. So

$$\frac{\mathrm{d}\hat{\boldsymbol{x}}_3}{\mathrm{d}t} = \left(\frac{\boldsymbol{L}}{I} - \Omega\hat{\boldsymbol{x}}_3\right)\times\hat{\boldsymbol{x}}_3 = \left(\frac{\boldsymbol{L}}{I}\right)\times\hat{\boldsymbol{x}}_3$$

    This is equivalent to $\hat{\boldsymbol{x}}_3$ rotating at a frequency L/I.
  - It turns out that we can interpret $\boldsymbol{\omega}$ as follows

$$\boldsymbol{\omega} = \overbrace{\frac{\boldsymbol{L}}{I}}^{\text{Motion of body around } L} - \overbrace{\Omega\hat{\boldsymbol{x}}_3}^{\text{Motion of body about its own axis}}$$

- **HEAVY SYMMETRIC TOP**
  - Here, we must define the **Euler angles** as follows



- The **total angular velocity** is then given by

$$\boldsymbol{\omega} = \overbrace{\dot{\psi}\hat{\boldsymbol{x}}_3}^{\text{Rotation of top}} + \overbrace{\dot{\theta}\hat{\boldsymbol{x}}_1 + \dot{\phi}\boldsymbol{z}}^{\text{Motion of top itself}}$$

  Which can be expressed in terms of the body-frames only:

$$\boldsymbol{\omega} = \dot{\psi}\hat{\boldsymbol{x}}_3 + \dot{\theta}\hat{\boldsymbol{x}}_1 + \dot{\phi}\left(\hat{\boldsymbol{x}}_3 \cos\theta + \hat{\boldsymbol{x}}_2 \sin\theta\right)$$

$$\boldsymbol{\omega} = \left(\dot{\psi} + \dot{\phi}\cos\theta\right)\hat{\boldsymbol{x}}_3 + \left(\dot{\phi}\sin\theta\right)\hat{\boldsymbol{x}}_2 + \dot{\theta}\hat{\boldsymbol{x}}_1$$

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Dynamics – Normal Modes

Introduction

- A **normal mode** of a **system** is an **oscillation** that has a **single frequency**.
- All the **more general oscillations** of the system can be expressed as **superpositions** of these normal modes.

General approach

- Consider a **system** defined by **generalised coordinates** q_i and acted on by forces F_i, moving in a **potential well** $U(\mathbf{x})$, and moving **elastically**.
- The **kinetic energy, T**, is then given by

$$T = \tfrac{1}{2}\sum\sum m_i \left|\dot{\boldsymbol{\xi}}_j(q_i)\right|^2$$

Where $\sum_i \boldsymbol{\xi}_j(q_i)$ is the **Cartesian coordinate** of the j^{th} part of the system, taken about an **equilibrium**, where all the $\boldsymbol{\xi}_j$ are 0. Expanding about that equilibrium:

$$\sum_i \boldsymbol{\xi}_j(q_i) = \sum_i \boldsymbol{\xi}_j(q_{i,\text{eq}}) + \left.\frac{\partial \boldsymbol{\xi}_j}{\partial q_i}\right|_{\text{eq}} q_i + \cdots$$

$$\sum_i \dot{\boldsymbol{\xi}}_j(q_i) \approx \sum_i \left.\frac{\partial \boldsymbol{\xi}_j}{\partial q_i}\right|_{\text{eq}} \dot{q}_i$$

And so:

$$T = \tfrac{1}{2}\sum\sum M_{ij}\dot{q}_i\dot{q}_j = \tfrac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}$$

Where

$$M_{ij} = \sum\sum m \left.\frac{\partial \boldsymbol{r}}{\partial q_i}\right|_{\text{eq}} \left.\frac{\partial \boldsymbol{r}}{\partial q_j}\right|_{\text{eq}}$$

- Consider the **potential energy**, about a **point of equilibrium** (ie: a **minimum in U**) at which all the $\boldsymbol{q_i}$ are chosen to be 0.

$$U(x) = U_0 + \underbrace{\sum \left.\frac{\partial U}{\partial q_i}\right|_{\text{Eq}} q_i}_{0\text{ since at a minimum}} + \sum\sum \frac{1}{2}\left.\frac{\mathrm{d}^2 U}{\mathrm{d}x_j \mathrm{d}x_i}\right|_{x_0} q_i q_j + \cdots$$

$$U(x) = U_0 + \frac{1}{2}\sum\sum K_{ij} q_i q_j + \cdots$$

$$U(x) = U_0 + \frac{1}{2}\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q}$$

- The **total energy** is then

$$E = U_0 + \frac{1}{2}\sum\sum M_{ij}\dot{q}_i\dot{q}_j + \frac{1}{2}\sum\sum K_{ij}q_iq_j$$
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}\sum\sum 2\dot{q}_i\left(M_{ij}\ddot{q}_j + K_{ij}q_j\right) = 0$$
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sum\sum \dot{q}_i\left(M_{ij}\ddot{q}_j + K_{ij}q_j\right) = 0$$

- [**Non rigorous argument**] – the equations of motion are then:

$$\sum\sum M_{ij}\ddot{q}_j + \sum\sum K_{ij}q_j = 0$$
$$\boxed{\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = 0}$$

- If we seek **normal modes** of the form $\boldsymbol{q}(t) = \boldsymbol{Q}e^{i\omega t}$, we get:

$$\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right)\boldsymbol{Q} = 0$$

  Non-trivial solutions only exist if

$$\boxed{\det\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right) = 0}$$

  This defines the ω^2 normal mode frequencies.

- In **practice**, the steps are:
  - Find the $\boldsymbol{K}$ and $\boldsymbol{M}$ matrices by writing them out in terms of the variables of the system, and comparing with

$$T = \tfrac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}} \qquad U = U_0 + \frac{1}{2}\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q}$$

    Both matrices **must** be **symmetric**.
  - Use the determinant method above.

Dynamics – Elasticity

Introduction

- **Hooke's Law** states that

$$\overbrace{\frac{F}{A}}^{\text{Stress}} = E\overbrace{\frac{\Delta l}{l}}^{\text{Strain}}$$

  Where
  - ***F*** is the **force** applied to a **block of material** over an **area *A***.
  - ***Δl*** is the **extension** of the **block *in the direction of F***.
  - ***l*** is the **original, relaxed length** of the block ***in that direction***.
  - ***E*** is the **Young's Modulus** of the material.
- Furthermore, it states that

$$\frac{\Delta w}{w} = -\sigma\frac{\Delta l}{l}$$

  Where ***Δw*** is the **length** of the block in **any direction perpendicular** to that of *l*.
- For an **isotropic material**, ***E*** and ***σ*** are all we need to define the elastic properties of the material.
- Since these equations are all **linear**, the **principle of superposition** applies. If we have **several forces**, the **displacements** will be the **sum of** the displacements with the forces acting **individually**.

Uniform Strain – the Bulk Modulus

- Consider a **rectangular** block in a **pressure tank**, say, with **identical stress *p*** on every face.
- Consider one direction – the **change in length *Δl*** in that direction is given by

$$\frac{\Delta l}{l} = \overbrace{-\frac{p}{E}}^{\substack{\text{Due to pressure}\\ \text{in that direction}}} + \overbrace{\sigma\frac{p}{E} + \sigma\frac{p}{E}}^{\substack{\text{Due to pressure in}\\ \text{other directions}}}$$

$$\frac{\Delta l}{l} = -\frac{1-2\sigma}{E}p$$

  The problem is **symmetrical**, so the value will be the same for **all directions**.

- Now, consider the **change in volume**

$$\frac{\Delta V}{V}=\frac{\Delta x}{x}+\frac{\Delta y}{y}+\frac{\Delta z}{z}$$

We therefore have

$$\frac{\Delta V}{V}=-3\frac{1-2\sigma}{E}p$$

- We can then define the **bulk modulus**

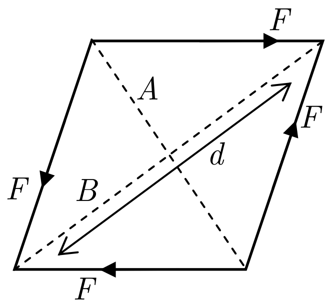
$$\boxed{K=\frac{E}{3(1-2\sigma)}}$$

Such that the **change of volume** as a result of the **stress *p*** is

$$p=-K\frac{\Delta V}{V}$$

Shear Strain – the Shear Modulus

- Consider a **cube** with **face area *A*** and with **shear forces** acting on it

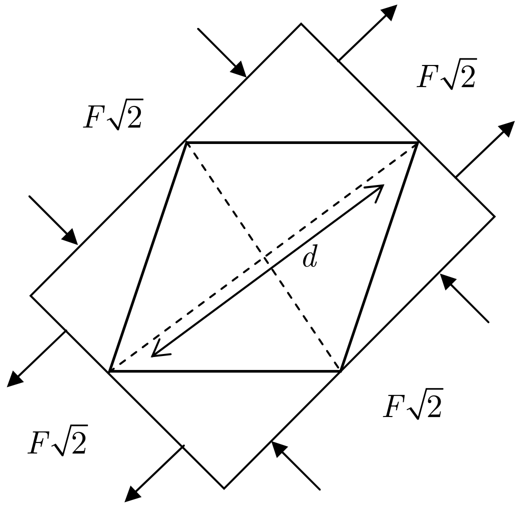


If cut the cube along the diagonals ***A*** and ***B***, we find that

  - There is a **stretch normal** to ***A***, of **magnitude** $\boldsymbol{F\sqrt{2}}$.
  - There is a **compression normal** to ***B***, of **magnitude** $\boldsymbol{F\sqrt{2}}$.

And each of these diagonal faces has **area** $A\sqrt{2}$.

- The lengthening of the diagonal d will therefore be equal to the lengthening of d in the following case:

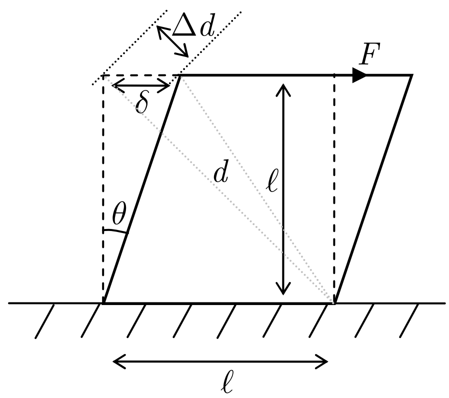


From above, this is given by:

$$\frac{\Delta d}{d} = \frac{1}{E}\frac{F\sqrt{2}}{A\sqrt{2}} + \sigma\frac{1}{E}\frac{F\sqrt{2}}{A\sqrt{2}}$$
$$\frac{\Delta d}{d} = \frac{1+\sigma}{E}\frac{F}{A}$$

By **symmetry**, the other diagonal is **shortened** by the same amount.

- It is often useful to have this as a function of the **twist angle**:



From this diagram, it is (reasonably) clear that

$$\delta = \Delta d\sqrt{2} \qquad d = \ell\sqrt{2}$$

Therefore

$$\theta \approx \frac{\delta}{l} = \frac{\Delta d\sqrt{2}}{l} = 2\frac{\Delta d}{d} = \frac{2(1+\sigma)}{E}\frac{F}{A}$$

- We therefore define the **shear modulus** as

$$\boxed{\mu = \frac{E}{2(1+\sigma)}}$$

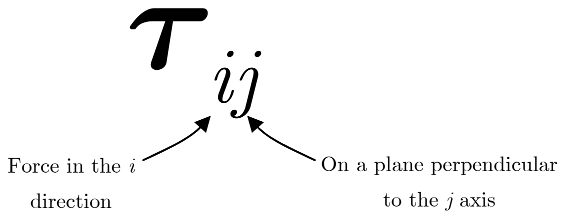
Such that

$$g = \mu\theta$$

Where g is the **shear stress** $= \boldsymbol{F/A}$.

Formal Definitions

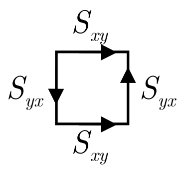
- **Stress**
    - Defined in terms of ***force/unit area*** transmitted across **planes** in the medium.
    - Requires a **tensor**. We define



    - We can then show that the force on **any arbitrary area element** is

$$\boldsymbol{F} = \boldsymbol{\tau}\,\mathrm{d}\boldsymbol{S}$$

    - The tensor must be **symmetric** – consider a **small cube** side dx. Because the cube must be in **equilibrium**, the forces on it are as follows:



The net **couple** on the cube is

$$\left(S_{xy} - S_{yx}\right)\mathrm{d}x$$

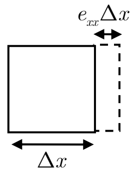
But there must be **no torque** on the cube, or it'd spin! So

$$S_{xy} = S_{yx}$$

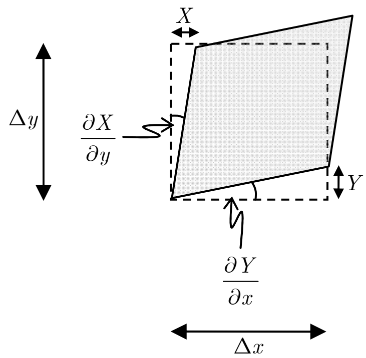
    - The stress tensor is **diagonal** for **suitable choices of axes**.
    - The stress in a solid material is therefore described by a **tensor field**.
- **Strain**
    - When a material is put under **strain**, a point (x, y, z) in it is moved to a point $(x + X, y + Y, z + Z)$.
    - The **derivatives** of these X, Y and Z contain **information** about the **strain**.
    - As we saw before, it's worth considering **two kinds of strain**
        - For the **normal strains**, we define:

$$e_{xx} = \frac{\partial X}{\partial x} \qquad e_{yy} = \frac{\partial Y}{\partial y} \qquad e_{zz} = \frac{\partial Z}{\partial z}$$

For example, if we consider stress **perpendicular** to the ***x*** **direction** in a **cube** initially of side Δx, it'll **increase** by $e_{xx}\Delta x$:



- Now, for the **shear stresses**, consider



[The expression for the angles are tricky to see – but consider that X is the **change** in x...] We then simply define

$$e_{xy} = e_{yx} = \tfrac{1}{2}\left(\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\right)$$

This ensures that if the block simply *rotates* (ie: $\partial Y / \partial y = \partial X / \partial x$), these strains are 0.

- So in general, we define

$$e_{ij} = \tfrac{1}{2}\left(\partial X_j / \partial x_i + \partial X_i / \partial x_j\right)$$

The i^{th} coordinate of a point in the material...

...will have changed by $e_{ij}D$, assuming that the j^{th} coordinate of that point from the origin is D.

So, for example

$$X = e_{xx}x + e_{xy}y + e_{xz}z$$

- The tensor is also **symmetric**, due to the $e_{xy} = e_{yx}$ condition.
- If the strains are **non-homogenous**, we sit down and cry.

- **The relation between them**

- **Each** component of e is related to *each* component of τ – this gives, overall, a **fourth-rank tensor of elasticity** relating the two:

$$\tau_{ij} = C_{ijkl} e_{kl}$$

(Using the summation convention).

- It looks like there are $9^2 = 81$ **coefficients** in C, and that **81 numbers** are therefore required to **define** the **elastic properties** of a material! However, we note that since S and e are symmetry, we must be able to swap ij and kl in C without changing a thing, so there can be at most 36 different coefficients.
- If the material is **isotropic**, though, C must be completely **frame-independent**. As such, we **must** be able to **express** it in terms of the tensor δ_{ij}. There are only **two** ways of doing this that are also **invariant** under $i \leftrightarrow j$ and $l \leftrightarrow k$, and so

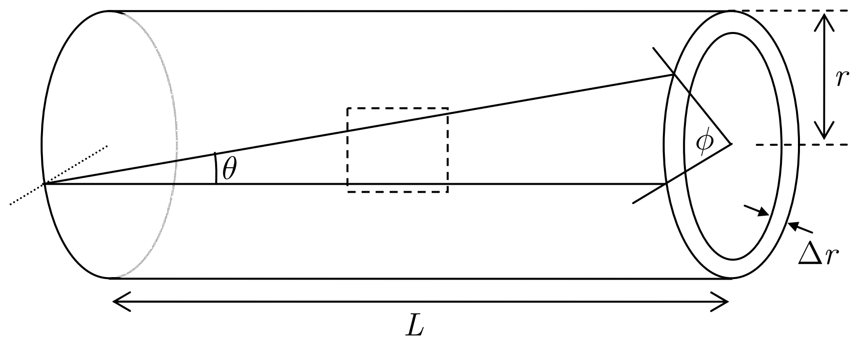
$$C_{ijkl} = \lambda\left(\delta_{ij}\delta_{kl}\right) + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

So an **isotropic material** only requires **two constants** (E and σ, for example). And we have

$$S_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

Examples – Statics

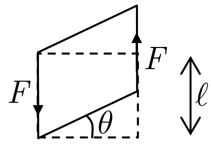
- ***Thin tube in torsion***
  - Consider a **thin tube** being **twisted** an angle ϕ



  - We first note that

$$\theta = \frac{r\phi}{l}$$

  - Next, consider a small square (dotted above) and its deformation as a result of the twist:



From the previous result:

$$\frac{F}{\ell \Delta r} = \mu\theta$$

$$F = \mu \frac{r\phi}{L} \ell \Delta r$$

- This force contributes a **torque $\Delta\tau$** to the rod

$$\Delta\tau = rF = \mu \frac{r^2\phi}{L} \ell \Delta r$$

- Considering these bits around the **whole rod**, so that $\ell \to 2\pi r$, we get

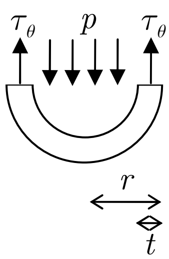
$$\tau = 2\pi\mu \frac{r^3 \Delta r}{L} \phi$$

- ***Wire in torsion***

  For a **wire**, we simply **integrate** the above from $r = 0$ to the total radius, giving

$$\tau = \mu \frac{\pi r^4}{2L} \phi$$

- ***Can under pressure***
  - Consider a **can** of **thickness *t*** with **closed ends** with an **internal pressure *p***.
  - Let the **tangential stress** in the walls be τ_θ, and consider *half* the can



The **forces** (= *stress* × *area*) must balance, so

$$\tau_\theta \times 2t = p \times 2r$$

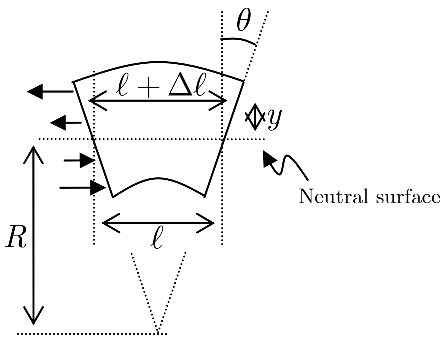
$$\tau_\theta = \frac{pr}{t}$$

  - Let the **axial stress** in the walls be σ_z, and consider one of the **ends**. By the same logic as above

$$\tau_z \times 2\pi r t = p \times \pi r^2$$

$$\tau_z = \frac{pr}{2t}$$

- ***Bent beam***
  - Consider a **beam** of length ***L***, held in a **bent** position.
  - We only consider **longitudinal strains** (valid for **small deflections** and **thin beams**).
  - Clearly, the bits at the **top** of the beam will be **stretched**, while those at the **bottom** will be **compressed**. Somewhere in between, there'll be a **neutral surface** – neither **stretched** nor **compressed**.
  - Consider a small segment length ℓ of the bent beam:



  - The **amount** of **stretching** and **compression** at any point is **proportional** to the **distance from the neutral surface, *y***. The **constant of proportionality** is ℓ / R. As such

$$\frac{\Delta \ell}{\ell} = \text{Strain} = \frac{y}{R}$$

  - Clearly, there'll be forces to the **left above** the neutral surface, and **vice versa**. We therefore have

$$\frac{\Delta F}{\Delta A} = E\frac{\Delta \ell}{\ell}$$

$$\Delta F = \frac{E}{R} y \Delta A$$

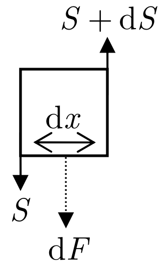
  - The **total torque produced** about the **neutral line** is given by

$$\boldsymbol{\tau} = \int_{\substack{\text{Cross}\\\text{section}}} y \,\mathrm{d}F$$

$$= \frac{E}{R}\int_{\substack{\text{Cross}\\\text{section}}} y^2 \,\mathrm{d}A$$

$$\boxed{B = \frac{EI}{R}}$$

- Now, consider a **beam loaded** with **weights** given by ***W(x)***, where ***W*** is the **force per unit length**. Consider the statics of a small **segment** of the beam:



Notes:

  - Due to the **bending moment**, some **vertical forces** are produced. Ignoring products of infinitesimal quantities, we can write, **at that point**

$$S\,\mathrm{d}x = \mathrm{d}B$$

$$B = \frac{EI}{R} = \int S\,\mathrm{d}x$$

  [Effectively, we're saying that due to the $\mathrm{d}S$ needed to balance $\mathrm{d}F$, the bending moment must change]

  - The **downwards loading force** needs to be **balanced** by a **difference** in the **upwards stress**

$$\mathrm{d}S = \mathrm{d}F = W\,\mathrm{d}x$$

$$S' = W$$

- Now, for **small deflections**

$$y'' = 1/R$$

- As such, we can conclude

$$\boxed{EIy'''' = W(x)}$$

- **Boundary conditions** for various cases are as follows
  - At a **free end**, ***S*** and ***B*** are clearly 0, and so $y'' = y''' = 0$.
  - At a **cantilevered end**, y and y' are given (usually 0).
- Finding y is then simply a question of solving that differential equation. **However**, there are a few **tricky points**
  - **All** forces must be considered when writing down ***W(x)***, including **reactions** at **contacts**. Most often, ***W*** will be a series of δ-functions.
  - Sign conventions:

- **Downwards $W \to$ positive**.
- The resulting y obtained is **downwards → positive**, because the way the radius of curvature is specified.
- However, be ***very*** careful – sometimes, the convention appears to be reversed because the bar curves downwards, and so $-1/R = y''$.

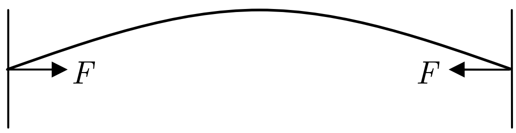
▪ Don't worry too much about boundary conditions for y''' – just integrate δ-functions from 0 to L (for a free end, this is fully justified). Remember that there'll often be a δ-functions at the *very end* of the range, which might help satisfy the boundary conditions.

▪ From then on, **boundaries** are just provided. Just also remember to make the y'', y' and y **continuous**.

▪ The **couple** provided by a **cantilever** can simply be worked out by evaluating $\boldsymbol{B = EIy''}$ at that point.

▪ It is sometimes easier to simply **write down** y'', the **bending moment** from **physical considerations**.

o The **Euler Strut** is a beam **buckled** between **two walls**:



If we take y **upwards**, then the **bending moment** on **any point** is

$$B = -Fy$$

$$y'' = -\frac{F}{EI}y$$

$$y = A\sin\left[x\sqrt{\frac{F}{EI}}\right]$$

Applying the boundary condition that $y = 0$ at $x = L$:

$$F = \frac{\pi^2 EI}{L^2}$$

This is **independent of displacement** (but only while $y'' = 1/R$ holds).

o The **Reciprocity Theorem** states that

"*The deflection at Q due to a load at P is the same as the deflection at P due to the same load at Q*"

To prove, say P_{PQ} means "the deflection at P due to the load at Q". Consider loading first P and then Q. The energy stored is

$$E = F\left[\frac{P_{PP}}{2} + \frac{P_{QQ}}{2} + P_{PQ}\right]$$

The same result must be applied the other way round, so

$$\boxed{P_{PQ} = P_{QP}}$$

Dynamics of Rigid Bodies

- Consider a **small volume** $\boldsymbol{V}$ of the material. It will have both **external** forces acting on it (eg: gravity) and **internal forces** (eg: elastic stresses).

$$\boldsymbol{F}_{\text{ext}} + \boldsymbol{F}_{\text{int}} = \int \rho \ddot{\boldsymbol{r}} \, \mathrm{d}V$$

- Every small particle in the volume experiences the **external** force, though, so $\boldsymbol{F}_{\text{ext}}$ is given by a **volume integral**.

$$\boldsymbol{F}_{\text{int}} = \int \overbrace{(-\boldsymbol{f}_{\text{ext}} + \rho\ddot{\boldsymbol{r}})}^{\text{Define this } = \boldsymbol{f}} \mathrm{d}V$$

$$\boldsymbol{F}_{\text{int}} = \int_V \boldsymbol{f} \, \mathrm{d}V$$

On the other hand, only the particles at the **edge** of the volume experience the **elastic** force from surrounding media, and so $\boldsymbol{F}_{\text{int}}$ is given by an **area integral**

$$\int_A \boldsymbol{f}_{\text{int}} \, \mathrm{d}A = \int_V \boldsymbol{f} \, \mathrm{d}V$$

- We have, however, defined that the force in the x-direction, say, is

$$\mathrm{d}F_x = \left(S_{xx}\boldsymbol{i} + S_{xy}\boldsymbol{j} + S_{xz}\boldsymbol{k}\right) \cdot \mathrm{d}\boldsymbol{A}$$

And so, taking only the x component of the integral above

$$\int_A \left(S_{xx}\boldsymbol{i} + S_{xy}\boldsymbol{j} + S_{xz}\boldsymbol{k}\right) \cdot \mathrm{d}\boldsymbol{A} = \int_V f_x \, \mathrm{d}V$$

- Using the Divergence Theorem on the LHS

$$\int_V \left(\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} + \frac{\partial S_{xz}}{\partial z}\right) \mathrm{d}V = \int_V f_x \, \mathrm{d}V$$

Removing the volume integrals (because this is true for any volume):

$$\boldsymbol{f}_i = \partial S_{ij} / \partial x_j$$

(Using the summation convention).

- Now, using $\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$ (isotropic material), we obtain

$$\boldsymbol{f} = (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u}$$

Where $\boldsymbol{u}$ is the internal displacement in the solid.