Electrical Circuits

Basic Principles

- The **electric current**, I, through a surface is the **rate of flow of charge**, q, through that surface. So:
$$I = \frac{\mathrm{d}q}{\mathrm{d}t} \qquad \text{and} \qquad Q = \int_0^t I \, \mathrm{d}t$$
If I is constant, the latter becomes $Q = It$.
- All current is assumed to be due to the motion of **positive charges**, and arrows are drawn in the direction in which **positive** charge would move.
- Currents in wires are driven by $\boldsymbol{E}$-fields [when a current is flowing in a conductor, electrostatic equilibrium no longer exists and the $\boldsymbol{E}$ field is not zero].
- In the absence of an $\boldsymbol{E}$ field, electrons move at high speeds in random directions. When an $\boldsymbol{E}$ field is applied, each electron is accelerated by a force $e\boldsymbol{E}$ and accelerates until it collides with a fixed ion of the lattice. In each collision, energy is given to the lattice. As a result, the lattice gains energy and heats up and each electron acquires a small average velocity called the **drift velocity**.
- The **potential difference** between two points in the circuit is defined as **the amount of electrical energy changed to other forms of energy when unit charge passes from one point to the other**. Thus, in general, if charge flows in a part of the circuit across which there is a potential difference of V, then the work done is given by
$$W = \int_0^Q V \, \mathrm{d}q = \int_0^t VI \, \mathrm{d}t$$
If V and I are constant, then $W = VIt$.
- Some conductors offer more resistance to the passage of current than others. The **resistance**, R, of a conductor is defined as
$$R = V / I$$
The interaction is mainly between the electrons and the crystal lattice, but also between electrons and defects in the lattice.
- **Batteries** and **generators** moves charges from a place of **low potential** to a place of **high potential**. In doing that, the battery does work. The battery

is said to produce an **electromotive force (EMF)** E, where E is the amount of energy converted to potential energy when unit charge passes through the battery or generator.

Kirchhoff's Laws

- Kirchhoff's **First Law** states that **the algebraic sum of all the currents meeting at a point is 0**. In other words

$$\sum I = 0$$

  Conventions:

  - Incoming current is positive
  - Outgoing current is negative

  This is a law of conservation of charge.

- Kirchhoff's **Second Law** states that the algebraic sum of the potential differences across the resistances in **any closed circuit** (or any closed loop of a circuit) is equal to the sum of the EMFs in that circuit. In other words

$$\sum E = \sum IR$$

  Conventions:

  - When moving across a resistor **in the direction of the current** (ie: along a potential drop), the IR product is **positive**.
  - When moving across a battery from the **negative** to the **positive** terminal (in other words, in the direction in which electrons are pumped **up**), the EMF is positive.
  - Treat **inductors** exactly like resistors – when moving across it *in the direction of the current*, the $L\, \mathrm{d}I/\mathrm{d}t$ product is *positive* on the non-EMF side.
  - For **capacitors**, the q/C term is **positive on the *IR* side** if we're moving from the +ve to the –ve side of the capacitor.
  - [These can all also be worked out using potential arrows!]

  This is a law of conservation of energy, because it states that when going round a loop, one must find oneself at the same potential one started off at.

Combinations of Resistors

- For resistors in **series**, use the fact that the total PD across the resistors is equal to the sum of the individual PDs, and then use $V = IR$.
- For resistors in **parallel**, use the fact that the total current across the parallel system is equal to the sum of the individual currents, and use $V = IR$.

Miscellaneous stuff…

- An ideal battery (or generator) is assumed to produce the same voltage across its terminals regardless of what current is drawn from it. In practise, the battery offers a resistance to current flow – the **internal resistance**, r, which behaves as a resistor in series with the battery. Thus, E decreases as the current drawn increases.

  If the resistance of the circuit attached to the battery is R, then the current flowing through the circuit is $I = E(R + r)$, so the p.d. across the rest of the circuit (= that across the battery) is
  $$V_{effective} = \frac{R}{R+r}E$$
  If R is very large, then I is very small and $V_{effective} \approx E$.
- Current flow is accompanied by the conversion of electrical energy into other forms. The rate at which this happens is the **power** of a device.

  We showed above that $\mathrm{d}W = V\, I\, \mathrm{d}t$. So:
  $$\frac{\mathrm{d}W}{\mathrm{d}t} = VI = \frac{V^2}{R} = I^2R$$
  The transfer maximum power from one part of a circuit to another, the impedances of the two parts must be the same.

Capacitors & Inductors

- Use Kirchhoff's second law in the sensible way.
- Capacitors
  - $Q = CV$, which gives starting conditions.

  - $I = \mathrm{d}q/\mathrm{d}t$ and $V = q/C$. This allows us to find a differential equation in terms of q.
  - We can also write $I = \mathrm{d}q/\mathrm{d}t = \mathrm{d}(CV)/\mathrm{d}t = C\ \mathrm{d}V/\mathrm{d}t$, which allows us to find a differential equation in terms of V.
  - The current can be found by finding an expression for q and differentiating.
  - The resulting expression is

    $$V = V_0 e^{-t/RC}$$

    The time taken for the voltage to drop by e is RC, and this is called the **time constant** of the circuit.
- Inductors
  - The potential difference across an inductor is proportional to the **rate of change of current** in the inductor:

    $$V = L\frac{\mathrm{d}I}{\mathrm{d}t}$$

    And this is in such a direction to **oppose the change in increase or decrease** of current.
  - For an LR circuit, the **time constant** is given by L/R.

Oscillation in Circuits

- In an LC circuit, we find SHM with $\omega^2 = 1/LC$.
- The capacitor is similar to a spring [$k = 1/C$], the inductor to a mass [$m = L$] and the resistor to a damper.
- The charge [and voltage across the capacitor] is similar to the displacement, the current to the velocity and the voltage across the inductor to the acceleration.
- The energy in an LC circuit is constant and, on average, shared equally between the capacitor and inductor.

Complex Impedance

- **Resistors**
  - Let's consider applying a voltage $V = V_0 e^{i\omega t}$ - the resulting current will be

$$I = \frac{V_0}{R} e^{i\omega t}$$

- The **impedance** of a resistor is therefore simply R.
- The instantaneous power supplied to a resistor is given by

$$P = VI = \frac{V_0^2}{R} \cos^2 \omega t = \frac{V_0^2}{R} \frac{1}{2}(1 + \cos 2\omega t)$$

  Over one cycle, the second term averages to 0. So, the mean power dissipated is

$$\overline{P} = \frac{1}{2} \frac{V_0^2}{R} = \frac{V_{rms}^2}{R}$$

  Where $V_{rms} = V_0 / \sqrt{2}$.

- **Capacitors**
  - **Capacitors** are designed to **store electric charge** – in an ideal capacitor, no current can flow *through* the space in the capacitor, and so there can be **no steady current** in the part of the circuit containing an ideal capacitor.
  - Let's consider applying a voltage $V = V_0 e^{i\omega t}$ - the resulting current will be

$$I = \dot{q} = \frac{\mathrm{d}}{\mathrm{d}t} CV = CV_0 i\omega e^{i\omega t} = CV_0 \omega e^{i\pi/2} e^{i\omega t} = CV_0 \omega e^{(\omega t + \pi/2)i}$$

  - We can then define the **complex impedance** of a capacitor as

$$Z = \frac{V}{I} = \frac{V_0 e^{i\omega t}}{i\omega C V_0 e^{i\omega t}} = \frac{1}{i\omega C}$$

  - The instantaneous power supplied to the capacitor is

$$P = -\omega C V_0^2 \cos \omega t \sin \omega t = -\tfrac{1}{2} \omega C V_0^2 \sin 2\omega t$$

    Over a half-cycle, this averages to 0 – not net power is supplied to the capacitor. It charges up for a quarter of a cycle, and release energy in the next quarter.

- **Inductor**
  - An **inductor** is a coil with (ideally) 0 resistance, that opposes any changes in the current flowing through it, such that $V_{across\ inductor} = L \frac{\mathrm{d}I}{\mathrm{d}t}$
  - Let's consider a voltage $V = V_0 e^{i\omega t}$ across the inductor.

$$V_0 e^{i\omega t} = L \frac{\mathrm{d}I}{\mathrm{d}t}$$

$$\Rightarrow I = \frac{V_0}{L} \int_0^t e^{i\omega t}\, \mathrm{d}t = \frac{V_0}{i\omega L} e^{i\omega t} = -i \frac{V_0}{\omega L} e^{i\omega t} = \frac{V_0}{\omega L} e^{(\omega t - \pi/2)i}$$

- We can therefore define the **impedance** of an inductor as
$$Z = \frac{V}{I} = i\omega L$$
- Again, looking at the power
$$P = \frac{V_0^2}{\omega L}\frac{1}{2}\sin 2\omega t$$
Therefore, like a capacitor, an inductor does not store energy.

- Complex impedances add like resistances.

Filters & Circuits

- The RC filter preferentially selects low frequencies.
- The LC filter preferentially selects a particular frequency.
- The series LRC filter is similar to the LC filter.
- The natural resonant frequency of LC and LRC circuits is
$$\omega_0 = \frac{1}{\sqrt{LC}}$$
At this point, the impedance has its lowest value, and is equal to R. [Note – this only approximates the resonance peak in *weakly* damped systems].
- The **power** dissipated in the resistor is given by
$$\bar{P} = \frac{1}{2}\left|I\right|^2 R = \frac{1}{2}\frac{V_0^2}{\left|Z\right|^2}R$$
At resonance, when the impedance is equal to R
$$\bar{P} = \frac{1}{2}\frac{V_0^2}{R}$$
- The **Bandwidth** is the distance between the **half-power points** of the circuit – ie: those points at which the power dissipated in the circuit is half the power dissipated at resonance.
$$P_{hp} = \frac{1}{2}P_{res} \Rightarrow \left|I_{hp}\right| = \frac{1}{\sqrt{2}}\left|I_{res}\right| \Rightarrow \left|Z_{hp}\right|^2 = 2\left|Z_{res}\right|^2 \Rightarrow \left|Z_{hp}\right|^2 = 2R^2$$
Making the approximation that $(R/L)^2 \ll \omega_0^2$, we find that
$$\text{Bandwidth} = \frac{R}{L}$$
The ratio of the resonant frequency to the bandwidth is called the **quality factor**:
$$Q = \frac{\omega_0}{\text{Bandwidth}} = \frac{\omega_0 L}{R}$$
And the higher the value of Q, the higher the resonance.

Handy tips

- $i = e^{i\pi/2}$ and $1/i = -i$.
- Never rationalise the denominator – work it out all the way through, and then use the fact that the magnitude of one complex number divided by the other is equal to one magnitude divided by the other, etc…
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