Chapter 2 + 3 – Relativity & Extra Dimensions

1. Stuff

- The UP indices are the real ones. The DOWN indices are missing a minus sign. So $\mathrm{d}x_0 = \left(-\mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3\right)$
- A three-ball, B^3 is surrounded by the two-sphere S^2. In general
$$\mathrm{vol}\left(S^{d-1}(R)\right) = R^{d-1}\mathrm{vol}\left(S^{d-1}\right) = R^{d-1}\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$$
$$\mathrm{vol}\left(B^d\right) = \frac{\pi^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)}$$
- In general
$$\int_{B_d} \nabla \cdot \boldsymbol{E}\,\mathrm{dV} = \int_{B^d} \rho\,\mathrm{dV}$$
$$\text{Flux of } \boldsymbol{E} \text{ across } S^{d-1} = q$$
For each extra dimension, we get an extra factor of $1/r$.
- $\left[G\right] = m^3 / kg\,s^2$, $\left[c\right] = ms^{-1}$ and $\left[\hbar\right] = kg\,m^2 / s$. $\ell_P = \sqrt{G\hbar / c^3}$
- $\boldsymbol{g} = -\nabla V$ and $\nabla^2 V^{(D)} = 4\pi G^{(D)}\rho_m$. Since the dimensions of the density change, the dimensions of G must change as well.
- We get $\left(\ell_P^{(D)}\right)^{D-2} = \dfrac{\hbar G^{(D)}}{c^3} = \left(\ell_P\right)^2 \dfrac{G^{(D)}}{G}$
- $\dfrac{G^{(D)}}{G} = V_C$, where V_c is the product of all the different characteristic lengths of the extra dimensions.

Chapter 4 – Nonrelativstic Strings

1. Introduction

- **Small oscillations** of a string imply that

$$\left|\frac{\partial y}{\partial x}\right| \ll 1$$

- Using $(\partial y / \partial x) \approx \tan\theta$, we can derive the fact that the motion of the string satisfies the **wave equation**

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2} = 0$$

  Disturbances along the string therefore move at a velocity

$$v_0 = \sqrt{T_0 / \mu_0}$$

- If each point on the string is oscillating sinusoidally and in phase, with $y(t,x) = y(x)\sin\left(\omega_n t + \phi\right)$, we can feed this into our wave equation, and find solutions for Dirichlet and Neumann boundary conditions
  - Dirichlet conditions give

$$y_n(x) = A_n \sin\left(\frac{n\pi x}{a}\right) \qquad \omega_n = \frac{n\pi}{a}\sqrt{\frac{T_0}{\mu_0}} \qquad n = 1,2,\cdots$$

  - Neumann conditions give

$$y_n(x) = A_n \cos\left(\frac{n\pi x}{a}\right) \qquad \omega_n = \frac{n\pi}{a}\sqrt{\frac{T_0}{\mu_0}} \qquad n = 0,1,\cdots$$

    This case clearly admits an extra mode of motion ($n = 0$) which corresponds to the string translating along the y-axis.

- More generally, the most general solution of this equation is given by

$$y(t,x) = h_+\left(x - v_0 t\right) + h_-\left(x + v_0 t\right)$$

  [This is the superposition of a wave travelling towards the left and one travelling towards the right].

- The functions for h_+ and h_- can be related using **initial** and/or **boundary conditions** – these can be of two forms
  - **Dirichlet** conditions specify the value of y.

  - **Neumann** conditions specify the value of a derivative of y with respect to position along the string.

- For strings with non-constant densities $\mu(x)$, the wave equation still applies, since it was derived by considering a *small* piece of string. The analysis leading to $y(x)$ involves solving a slightly more complex differential equation.

2. Lagrangian Mechanics

- The **lagrangian** for a system is defined by
$$L = T - V$$
Where T is the kinetic energy and V is the potential energy of the system.
- The **action** for a given path $\mathcal{P}$ the particle might take is defined as
$$S = \int_{\mathcal{P}} L(t)\,\mathrm{d}t$$
Hamilton's Principle states that the path $\mathcal{P}$ which a system actually takes is one for which the action S does not change to first order when $\mathcal{P}$ is varied infinitesimally.
- Usually, the path is parameterised by time as $x(t)$, and the perturbed path takes the form $x(t) + \delta x(t)$. The integral then takes the form
$$S = \int_{t_i}^{t_f} L(x, \dot{x})\,\mathrm{d}t$$
We usually only consider variations to the path that are fixed at the start and end of the motion, such that $\delta x(t_i) = \delta x(t_f) = 0$.

3. Lagrangian Mechanics for a nonrelativistic string

- For a string, the kinetic energy is the sum of the kinetic energies of all the infinitesimal segments that comprise the string:
$$T = \int_0^a \frac{1}{2}\mu_0 \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x$$
The potential energy comes from the work that must be done to stretch each individual segment of the string. If each small part of the string is stretched by an amount $\mathrm{d}\ell$, then the work done stretching it is $T_0\,\mathrm{d}\ell$ and

$$V = \int_{\text{string}} T_0 \,\mathrm{d}\ell$$

For a small segment $\mathrm{d}x$ of string that is displaced $\mathrm{d}y$ vertically, we have

$$V = \int_{\text{string}} T_0 \left(\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} - \mathrm{d}x \right) \Rightarrow V = \int_0^a T_0 \left[\mathrm{d}x \sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2} - \mathrm{d}x \right] \mathrm{d}x$$

$$V = \int_0^a \frac{1}{2} T_0 \left(\frac{\partial y}{\partial x} \right)^2 \mathrm{d}x$$

Where the small-oscillation approximation was used to ignore higher terms in the Taylor expansion.

- The string Lagrangian is therefore

$$L(t) = \int_0^a \frac{1}{2}\mu_0 \left(\frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} T_0 \left(\frac{\partial y}{\partial x} \right)^2 \mathrm{d}x = \int_0^a \mathcal{L} \,\mathrm{d}x$$

Where $\mathcal{L}$ is known as the **Lagrangian Density**.

- Varying the action gives

$$\delta S = \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x \, \mu_0 \dot{y} \frac{\partial}{\partial t}(\delta y) - T_0 y' \frac{\partial}{\partial x}(\delta y)$$

$$\delta S = \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x \frac{\partial}{\partial t}\left(\mu_0 \dot{y} \delta y \right) - \mu_0 \ddot{y} \delta y - \frac{\partial}{\partial x}\left(T_0 y' \delta y \right) + T_0 y'' \delta y$$

$$\delta S = \cancel{\int_0^a \mathrm{d}x \left[\mu_0 \dot{y} \delta y \right]_{t_i}^{t_f}} - \int_{t_i}^{t_f} \mathrm{d}t \left[T_0 y' \delta y \right]_0^a + \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x \left[T_0 y'' - \mu_0 \ddot{y} \right] \delta y$$

There are three terms in this action

- The first one corresponds to insisting our path does not vary at its time end-points.
- The third involves the domain $x \in (0,a), t \in \left(t_i, t_f\right)$, and so we can simply set the coefficient to 0 and recover the wave equation $T_0 y'' - \mu_0 \ddot{y} = 0$.
- The second term involves the **end** of the strings (**boundary conditions**). There are two ways to make this term vanish
  - **Set** $y' = 0$ **at the ends** – this corresponds to **Neumann** boundary conditions.
  - **Set** $\delta y = 0$ **at the ends** ($\Rightarrow \dot{y} = 0$ at the ends, since t is what is being varied in this integral) – this corresponds to **Dirichlet** boundary conditions.

- Note that the momentum in the y-direction along the string is given by
$$p_y = \int_0^a \mu_0 \frac{\partial y}{\partial t} \mathrm{d}x$$
And
$$\frac{\mathrm{d}}{\mathrm{d}t} p_y = \int_0^a \mu_0 \frac{\partial^2 y}{\partial t^2} \mathrm{d}x = T_0 \left[\frac{\partial y}{\partial x}\right]_{x=0}^{x=a}$$
Thus, we see this is only conserved for Neumann boundary conditions.
- It is useful to define the following quantities
$$\mathcal{P}^t \equiv \frac{\partial \mathcal{L}}{\partial \dot{y}} \qquad\qquad \mathcal{P}^x \equiv \frac{\partial \mathcal{L}}{\partial y'}$$
We can then vary the action as follows
$$S = \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathcal{L}\, \mathrm{d}x$$
$$\delta S = \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \left[\frac{\partial \mathcal{L}}{\partial \dot{y}} \delta \dot{y} + \frac{\partial \mathcal{L}}{\partial y'} \delta y'\right] \mathrm{d}x$$
$$\delta S = \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \left[\mathcal{P}^t \delta \dot{y} + \mathcal{P}^x \delta y'\right] \mathrm{d}x$$
$$\delta S = \int_0^a \left[\mathcal{P}^t \delta y\right]_{t_i}^{t_f} \mathrm{d}x + \int_{t_i}^{t_f} \left[\mathcal{P}^x \delta y\right]_0^a \mathrm{d}t - \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x \left[\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x}\right] \delta y$$
Which gives us an equation of motion which looks like
$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0$$
A few points of interest regarding this formulation
    - $\mathcal{P}^t$ is the momentum density along the string in the y-direction. This is because $\dot{y}$ is a **velocity** in the y direction, and so $\partial L / \partial y$ is the conjugate momentum in that direction.
    - The Neumann boundary condition implies $\mathcal{P}^x = 0$, and the Dirichlet boundary condition implies $\mathcal{P}^t = 0$.

4. Practical matters

- Tips for solving the wave equation using $y(t,x) = h_+\left(x - v_0 t\right) + h_-\left(x + v_0 t\right)$
    - First apply all boundary conditions, and see what they imply about the *single* variable functions h_+ and h_-. These tell us things at the **boundary** of the domain concerned.

- It is useful to remember that

$$\frac{\partial}{\partial x} h(x + vt) = h' \qquad \frac{\partial}{\partial y} h(x + vt) = v\dot{h}$$

- It is also useful to note that if $U = x + vt$ and $V = x - vt$, then

$$\frac{\mathrm{d}U}{\mathrm{d}V} = \frac{\mathrm{d}U}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}V} + \frac{\mathrm{d}U}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}V} = 1 - \frac{v}{v} = 0$$

So the two variables are indeed independent of each other.

- Tips for varying an action…
  - Vary terms as if you were differentiating them. If in doubt, substitute $x \leftarrow x + \delta x$ and simply, using Taylor series if any functions are involved.
  - It is a useful property that

$$\delta\left(\frac{\partial x}{\partial t}\right) = \frac{\partial}{\partial t}\left(\delta x\right)$$

  - If any derivatives of δx appear in the variation of S, use the strategy exemplified in the following example

$$\int \dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\delta x\,\mathrm{d}t = \int \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}\delta x\right)\mathrm{d}t - \int \delta x \frac{\mathrm{d}\dot{x}}{\mathrm{d}t}\mathrm{d}t$$

$$\int \dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\delta x\,\mathrm{d}t = \cancel{\left[\dot{x}\delta x\right]_{t_i}^{t_f}} - \int \delta x \frac{\mathrm{d}\dot{x}}{\mathrm{d}t}\mathrm{d}t$$

The first term vanishes because $\delta x(t_i) = \delta x(t_f) = 0$.

- The **canonical momentum** for a given Lagrangian is

$$\boldsymbol{p} = \frac{\partial L}{\partial\left(\dfrac{\partial \boldsymbol{x}}{\partial t}\right)}$$

And the **Hamiltonian** is given by

$$H = \left(\boldsymbol{p}\cdot\frac{\partial \boldsymbol{x}}{\partial t}\right) - L$$

When trying to find these for Hamiltonians including space-time vectors, it often helps to remove the time parts in dot products by explicitly multiplying them out.

If the Lagrangian does not depend explicitly, then the Hamiltonian is identified with the energy and is constant.

5. Questions remaining on this chapter

- For strings of non-constant densities, does the $h_+ + h_-$ solution still work?
- In problem 4.6, where does the m^2 come from?
- In problem 4.6, how do we do part (a)?

Chapter 5 – Relativistic Point Particle

1. The Relativistic Lagrangian

- By requiring the Lagrangian for a relativistic particle to be Lorentz invariant and dimensionally consistent, we find
$$S = -mc\int_{\mathcal{P}} \mathrm{d}s$$
For an observer moving with the particle, we have
$$S = -mc^2\int_{t_i}^{t_f} \mathrm{d}t\sqrt{1-\frac{v^2}{c^2}}$$
Which we can show is, indeed, kinetic minus potential energy.
- We can **parameterize** the worldline in terms of a parameter τ. This means that we have expressions for the coordinates in terms of τ: $x^\mu = x^\mu(\tau)$. Note that even the time coordinate is parameterized. We then can then use $\mathrm{d}s^2 = -\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ to write
$$\mathrm{d}s^2 = -\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\mathrm{d}\tau\right)^2$$
Which gives
$$S = -mc\int_{\tau_i}^{\tau_f} \mathrm{d}\tau\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}$$
We can show that this expression is **manifestly parameterisation invariant** by noting that if we choose a new parameter τ', then
$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau'}\frac{\mathrm{d}\tau'}{\mathrm{d}\tau}$$
- We can vary this action to find the equation of motion
$$\delta S = -mc\int \delta\left(\mathrm{d}s\right)$$
And
$$\mathrm{d}s^2 = -\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\mathrm{d}\tau\right)^2$$
$$2\mathrm{d}s\,\delta\left(\mathrm{d}s\right) = -2\eta_{\mu\nu}\frac{\mathrm{d}\left(\delta x^\mu\right)}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\mathrm{d}\tau^2\right)$$
$$\delta\left(\mathrm{d}s\right) = -\frac{\mathrm{d}\left(\delta x^\mu\right)}{\mathrm{d}\tau}\frac{\mathrm{d}x_\mu}{\mathrm{d}s}\left(\mathrm{d}\tau^2\right)$$

Feeding this back in and remembering that $p_\mu = mc\left(\mathrm{d}x_\mu / \mathrm{d}s\right)$ gives

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = 0$$

2. Relativistic Particle with Electric Charge

- If a particle is charged and in a region in which magnetic fields are present, the equation of motion is

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = \frac{q}{c} F_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}s}$$

- The action that leads to this equation of motion is

$$S = \frac{q}{c} \int_{\mathcal{P}} \mathrm{d}\tau \left[A_\mu \left(x(\tau)\right) \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}(\tau) \right]$$

- This term needs to be added to the relativistic Lagrangian found above to obtain the total Lagrangian for a charged particle in electromagnetic fields.

3. Practical Matters

- The current density from a single point particle of charge q moving through $D = d + 1$-dimensional space-time with $x^\mu(\tau) = \left(x^0(\tau), \boldsymbol{x}(\tau)\right)$ is given by

$$j^0(\boldsymbol{x}, t) = qc\delta^d\left(\boldsymbol{x} - \boldsymbol{x}(t)\right)$$
$$\boldsymbol{j}(\boldsymbol{x}, t) = q\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\delta^d\left(\boldsymbol{x} - \boldsymbol{x}(t)\right)$$

This can be written as

$$j^\mu(t, \boldsymbol{x}) = qc\int \mathrm{d}\tau \left[\delta^D\left(x - x(\tau)\right) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}(\tau) \right]$$

By noting that we can re-write this as

$$j^\mu(t, \boldsymbol{x}) = qc\int \mathrm{d}\tau \left[\delta^d\left(\boldsymbol{x} - \boldsymbol{x}(\tau)\right) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}(\tau) \right] \delta\left(x^0 - x^0(\tau)\right)$$

And changing variables to $X^0 = x^0(\tau) \Rightarrow \dfrac{\mathrm{d}X^0}{\mathrm{dx}^0 / \mathrm{d}\tau} = \mathrm{d}\tau$

$$j^\mu(t,\boldsymbol{x}) = qc\int \frac{\mathrm{d}X^0}{\mathrm{dx}^0 \,/\, \mathrm{d}\tau}\left[\delta^d\left(\boldsymbol{x}-\boldsymbol{x}(\tau)\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}(\tau)\right]\delta\left(x^0 - X^0\right)$$

$$j^\mu(t,\boldsymbol{x}) = qc\frac{1}{\mathrm{dx}^0 \,/\, \mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\delta^d\left(\boldsymbol{x}-\boldsymbol{x}(\tau)\right)\Big|_{x^0(\tau)=x^0}$$

$$j^\mu(t,\boldsymbol{x}) = qc\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0}\delta^d\left(\boldsymbol{x}-\boldsymbol{x}(\tau)\right)\Big|_{x^0(\tau)=x^0}$$

Which is precisely as above.

- It is sometimes useful to write

$$A_\mu\left(x(\tau)\right) = \int \delta\left(x - x(\tau)\right)A_\mu(x)\,\mathrm{d}^D x$$

4. Questions remaining on this chapter

- Don't understand the second component of the equation for current density above.
- In problem 5.3, why did we pick out x^0 in particular to separate, and not the other ones?
- Not entirely sure I'm confident with the $\delta\left(x - x(\tau)\right)$ notation. Shouldn't there be indices somewhere.

Chapter 6 – Relativistic Strings

1. Area Functionals

- Any surface can be characterised by two parameters – call them ξ^1 and ξ^2. Any point on the surface is then described by
$$\boldsymbol{x}\left(\xi^1,\xi^2\right) = \left(x^1(\xi^1,\xi^2), x^2(\xi^1,\xi^2), x^3(\xi^1,\xi^2)\right)$$
- Consider a rectangle of sides $\mathrm{d}\xi^1$ and $\mathrm{d}\xi^2$ in the image space. The corresponding "side-vectors" on our surface are
$$\mathrm{d}\boldsymbol{v}_1 = \frac{\partial \boldsymbol{x}}{\partial \xi^1}\mathrm{d}\xi^1 \qquad \mathrm{d}\boldsymbol{v}_2 = \frac{\partial \boldsymbol{x}}{\partial \xi^2}\mathrm{d}\xi^2$$
Using the formula for the area of a parallelogram
$$\mathrm{d}A = \left|\mathrm{d}\boldsymbol{v}_1\right|\left|\mathrm{d}\boldsymbol{v}_2\right|\left|\sin\theta\right|$$
$$\mathrm{d}A = \mathrm{d}\xi^1\mathrm{d}\xi^2\sqrt{\left(\frac{\partial \boldsymbol{x}}{\partial \xi^1}\cdot\frac{\partial \boldsymbol{x}}{\partial \xi^1}\right)\left(\frac{\partial \boldsymbol{x}}{\partial \xi^2}\cdot\frac{\partial \boldsymbol{x}}{\partial \xi^2}\right) - \left(\frac{\partial \boldsymbol{x}}{\partial \xi^1}\cdot\frac{\partial \boldsymbol{x}}{\partial \xi^2}\right)^2}$$
- This can be neatly summarised as
$$\mathrm{d}A = \mathrm{d}\xi^1\mathrm{d}\xi^2\sqrt{g}$$
Where $g \equiv \det g_{ij}$, where
$$g_{ij} = \frac{\partial \boldsymbol{x}}{\partial \xi^i}\cdot\frac{\partial \boldsymbol{x}}{\partial \xi^j}$$
is called the **induced metric**, such that the length $\mathrm{d}s$ of a vector on our surface is given by
$$\mathrm{d}s^2 = g_{ij}(\xi)\,\mathrm{d}\xi^i\,\mathrm{d}\xi^j$$

2. Spacetime surfaces

- For a **spacetime surface**, we use a similar device, and we call the two parameters special names – τ and σ. The mapping functions $X^\mu(\tau,\sigma)$ give us every coordinate on our spacetime surface in terms of these parameters.
- It so happens that the proper area of our spacetime surface is given by

$$A = \int \mathrm{d}\tau \, \mathrm{d}\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \tau}\right)\left(\frac{\partial X^\nu}{\partial \sigma}\frac{\partial X_\nu}{\partial \sigma}\right)}$$

$$A = \int \mathrm{d}\tau \, \mathrm{d}\sigma \sqrt{\left(\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2\left(\frac{\partial X}{\partial \sigma}\right)^2}$$

- To prove that the quantity under the square root must be positive, we consider the set of tangent vectors to the space-time surface given by

$$v^\mu(\lambda) = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma}$$

  We then take the dot product of v with itself, and since we require both timelike and spacelike tangent vectors to the string, this must take both positive and negative values. Thus, the discriminant of the resulting quadratic must be possible, and this gives us the proof we seek.
- If the quantity under the square root is 0, then the quadratic can only be positive or 0 – in other words, there are only spacelike or null vectors at that point on the string.

3. Nambu-Goto String Action

- Requiring the string action to be Lorentz Invariant and dimensionless gives us a string action of the following form

$$S = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \sqrt{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\right)^2\left(X'\right)^2}$$

$$S = \int_{\tau_i}^{\tau_f} L \, \mathrm{d}\tau = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \, \mathcal{L}(\dot{X}^\mu, X^{\mu\prime}) \qquad \mathcal{L} = -\frac{T_0}{c}\sqrt{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\right)^2\left(X'\right)^2}$$

$$S = -\frac{T_0}{c}\int \mathrm{d}\tau \, \mathrm{d}\sigma \sqrt{-\gamma} \qquad \gamma = \det\left(\gamma_{\alpha\beta}\right) \qquad \gamma_{\alpha\beta} = \begin{pmatrix} \left(\dot{X}\right)^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & \left(X'\right)^2 \end{pmatrix}$$

  Where a dot denotes a derivative with respect to τ, and a prime denotes a derivative with respect to σ.

4. Equations of motion

- We can vary this action

$$\delta S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \left[\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \frac{\partial\left(\delta X^\mu\right)}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} \frac{\partial\left(\delta X^\mu\right)}{\partial \sigma} \right]$$

$$\delta S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \left[\mathcal{P}_\mu^\tau \frac{\partial\left(\delta X^\mu\right)}{\partial \tau} + \mathcal{P}_\mu^\sigma \frac{\partial\left(\delta X^\mu\right)}{\partial \sigma} \right]$$

$$\delta S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[\delta X^\mu \mathcal{P}_\mu^\sigma\right]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma\, \delta X^\mu \left[\frac{\partial \mathcal{P}_\mu^\tau}{\partial \tau} + \frac{\partial \mathcal{P}_\mu^\sigma}{\partial \sigma} \right]$$

Where $\mathcal{P}_\mu^\tau$ and $\mathcal{P}_\mu^\sigma$ are given by horribly complicated formulae…

$$\mathcal{P}_\mu^\tau \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{\left(\dot{X} \cdot X'\right) X'_\mu - \left(X'\right)^2 \dot{X}_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 \left(X'\right)^2}}$$

$$\mathcal{P}_\mu^\tau \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{\left(\dot{X} \cdot X'\right) \dot{X}_\mu - \left(\dot{X}\right)^2 X'_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 \left(X'\right)^2}}$$

- Since the action must vanish, we require:

$$\frac{\partial \mathcal{P}_\mu^\tau}{\partial \tau} + \frac{\partial \mathcal{P}_\mu^\sigma}{\partial \sigma} = 0$$

- The first term above is a list that contains *two* terms for each index μ. This is a total of $2D$ conditions, each of which must be satisfied. There are two ways to satisfy each condition [σ^* denotes the σ -coordinate of an endpoint]:
    - **Dirichlet boundary conditions**: $\boxed{\dot{X}^\mu\left(\tau,\sigma^*\right) = 0}$. This can apply for all indices except for $\mu = 0$, because time (of course) varies with τ. In general, these conditions are best expressed as X = ?. ***Most importantly*** this condition does <u>not</u> need to be checked explicitly. It is automatically satisfied if the equation of motion holds and if $\mathcal{P}_\mu^\sigma(\tau,\sigma^*) = 0$ is not also satisfied.
    - **Free endpoint condition**: $\boxed{\mathcal{P}_\mu^\sigma(\tau,\sigma^*) = 0}$. This sets no constraint on the variation $\delta X^\mu(\tau,\sigma^*)$, and allows the end of the string to do whatever it needs to make the variation vanish. This must apply for time, and so $\mathcal{P}_0^\sigma(\tau,\sigma_1) = \mathcal{P}_0^\sigma(\tau,0) = 0$.

- For the end of a string to satisfy a Dirichlet boundary condition, it must be fixed to some object.
  - A **D*p*-brane** is an object with p spatial dimensions.
  - The ends of the string must be fixed on the **D*p*-brane**. So for every μ describing a direction *orthogonal* to the brane, we must have $X^\mu = 0$.
  - When open string endpoints have free boundary conditions along *all* directions, we have a **space-filling D-brane**.
- For the end of a string to be "free", we must have that

$$\mathcal{P}^{\sigma 0} = -\frac{T_0}{c}\frac{\frac{\partial \boldsymbol{X}}{\partial s}\cdot\frac{\partial \boldsymbol{X}}{\partial t}}{\sqrt{1-\frac{v_\perp^2}{c^2}}} = 0$$

  [Where we have used an expression derived later, using the static gauge]. This implies that
  - $\frac{\partial \boldsymbol{X}}{\partial s}\cdot\frac{\partial \boldsymbol{X}}{\partial t} = 0$, which implies that the velocity of the end-point is transverse to the string (since $\partial \boldsymbol{X} / \partial s$ is a vector parallel to the string).
  - Using the above fact, we can write $\vec{\mathcal{P}} = -T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\frac{\partial \boldsymbol{X}}{\partial s} = 0$, and this implies that at the endpoints, $v^2 = c^2$.

5. The Static Gauge – Parameterising time

- The **static gauge** is a choice of parameterisation that results in lines of constant τ being "static strings". In fact, we declare that for any point on the world-sheet, $\tau = t$, or

$$X^0\left(\tau,\sigma\right) \equiv ct\left(\tau,\sigma\right) = c\tau$$

- This choice of gauge implies that

$$X'^\mu = \left(\frac{\partial X^0}{\partial \sigma},\frac{\partial \boldsymbol{X}}{\partial \sigma}\right) = \left(0,\frac{\partial \boldsymbol{X}}{\partial \sigma}\right) \qquad \dot{X}^\mu = \left(\frac{\partial X^0}{\partial \tau},\frac{\partial \boldsymbol{X}}{\partial \tau}\right) = \left(c,\frac{\partial \boldsymbol{X}}{\partial \tau}\right)$$

  We have successfully separated time and space.

6. Tension and Energy of a Stretched String

- Consider a string stretched between points where $x^1 = 0,\ a$, with all other coordinates equal to 0. We say that the end are at $\left(0,\boldsymbol{0}\right)$ and $\left(a,\boldsymbol{0}\right)$. We then have that

$$X^\mu = \left(ct, f(\sigma), 0, \cdots\right)$$

Where $f(0) = 0$ and $f(\sigma_1) = a$ is the σ parameterisation of the string, and we have used the static gauge. We require $f' > 0$ to ensure that every point on the string has a unique σ coordinate.

- We then have

$$\dot{X}^\mu = \left(c, 0, \boldsymbol{0}\right) \qquad X'^\mu = \left(0, f', \boldsymbol{0}\right)$$

And so

$$\dot{X}\cdot\dot{X} = -c^2 \qquad X'\cdot X' = \left(f'\right)^2 \qquad \dot{X}\cdot X' = 0$$

- The action them becomes

$$S = -T_0\int_{t_i}^{t_f}\mathrm{d}t\int_0^{\sigma_1}\mathrm{d}\sigma\,\frac{\mathrm{d}f}{\mathrm{d}\sigma} = \int_{t_i}^{t_f}\left(-T_0 a\right)\mathrm{d}t$$

As expected, this does not depend on the way we parameterised σ.

- Now, since our string has no kinetic energy, we know that $L = -V$, and that S is the integral of that quantity. As such

$$V = T_0 a$$

This result is consistent with the observation that we are dealing with a "massless" string – all its "mass" comes from the energy expended to stretch it ($= T_0 a$). Indeed

$$\mu_0 c^2 = \frac{V}{a} = T_0 \Rightarrow \boxed{\mu_0 = \frac{T_0}{c^2}}$$

This confirms the identification of T_0 with the string tension, and confirms the negative sign in front of our action.

- We can also check that this configuration satisfies the equation of motion by using the horrible formulae for $\mathcal{P}^\tau$ and $\mathcal{P}^\sigma$ above, and it does indeed.

7. Action in terms of transverse velocity

- Defining a string velocity as $\partial \boldsymbol{X} / \partial t$ is too simplistic, because it depends on our choice of σ parameterisation (indeed, the vector $\partial \boldsymbol{X} / \partial t$ is parallel to lines of constant σ, since $t \propto \tau$).
- It turns out that the longitudinal velocity along the string is not physically meaningful, because it necessarily depends on σ parameterisation.
- Instead, we define a **transverse velocity**, $v_\perp$, which is (more or less) the velocity perpendicular to the string at any given time.
- To define it, we first note that if $s(\sigma)$ is a function measuring the length along the string at a given time, then $\mathrm{d}s = \left|\mathrm{d}\boldsymbol{X}\right|$ and $\partial \boldsymbol{X} / \partial s$ is a **unit vector tangent to the string**.
- Now, we note that the perpendicular component of a vector $\boldsymbol{v}$ to a direction $\hat{\boldsymbol{n}}$ is given by $\boldsymbol{v} - \left(\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\right)\hat{\boldsymbol{n}}$. As such, the component of the velocity $\partial \boldsymbol{X} / \partial t$ perpendicular to the string is

$$\boldsymbol{v}_\perp = \frac{\partial \boldsymbol{X}}{\partial t} - \left(\frac{\partial \boldsymbol{X}}{\partial t}\cdot\frac{\partial \boldsymbol{X}}{\partial s}\right)\frac{\partial \boldsymbol{X}}{\partial s}$$

$$\boldsymbol{v}_\perp^2 = \left(\frac{\partial \boldsymbol{X}}{\partial t}\right)^2 - \left(\frac{\partial \boldsymbol{X}}{\partial t}\cdot\frac{\partial \boldsymbol{X}}{\partial s}\right)^2$$

- We can then express the action in terms of this transverse velocity

$$S = -T_0\int \mathrm{d}t\int_0^{\sigma_1}\mathrm{d}\sigma\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)\sqrt{1-\frac{v_\perp^2}{c^2}}$$

With

$$L = -T_0\int \mathrm{d}s\sqrt{1-\frac{v_\perp^2}{c^2}}$$

In this form, the Lagrangian looks all nice and well – $T_0\mathrm{d}s$ is the rest energy, and we multiply it by a relativistic factor.

- From these, we can derive the following relations

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2}\frac{\left(\frac{\partial \boldsymbol{X}}{\partial s}\cdot\frac{\partial \boldsymbol{X}}{\partial t}\right)\dot{X}^\mu + \left(c^2 - \left(\frac{\partial \boldsymbol{X}}{\partial t}\right)^2\right)\frac{\partial X^\mu}{\partial s}}{\sqrt{1-\frac{v_\perp^2}{c^2}}} \qquad \mathcal{P}^{\sigma 0} = -\frac{T_0}{c}\frac{\frac{\partial \boldsymbol{X}}{\partial s}\cdot\frac{\partial \boldsymbol{X}}{\partial t}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}$$

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{\dot{X}^\mu - \left(\frac{\partial \boldsymbol{X}}{\partial s}\cdot\frac{\partial \boldsymbol{X}}{\partial t}\right)\frac{\partial X^\mu}{\partial s}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}$$

8. String endpoints

- We remember that at the endpoint, $\mathcal{P}^{\sigma 0} = 0$, which implies that $\frac{\partial \boldsymbol{X}}{\partial s}\cdot\frac{\partial \boldsymbol{X}}{\partial t} = 0$.
- This implies [see next chapter]

$$\boldsymbol{v}_\perp = \partial \boldsymbol{X} / \partial t$$

- We can use this to simply the expression for $\mathcal{P}^{\sigma\mu}$ at the endpoints. We get

$$\mathcal{P}^{\sigma\mu} = -T_0\sqrt{1-\frac{v^2}{c^2}}\frac{\partial X^\mu}{\partial s}$$

For the space coordinates

$$\vec{\mathcal{P}}^{\sigma} = -T_0\sqrt{1-\frac{v^2}{c^2}}\frac{\partial \boldsymbol{X}}{\partial s} = 0$$

Since $\partial \boldsymbol{X} / \partial s$ is a unit vector, we must have $v = c$.

9. Practical matters

- In some examples, it is obvious what $v_\perp$. For example, in a circular string that remains circular and just shrinks and grows, $v_\perp = \mathrm{d}R / \mathrm{d}t$.

10. Questions remaining on this chapter

- I don't get the reparametrisation invariant bit at the bottom of page 111.

Chapter 7 – String Parameterisation and Classical Motion

1. Parameterising the String

- We choose a paramaterisation in which lines of constant σ are perpendicular to the string (ie: to lines of constant t). Thus
$$\frac{\partial \boldsymbol{X}}{\partial \sigma} \cdot \frac{\partial \boldsymbol{X}}{\partial t} = 0$$
In this case, since $\partial \boldsymbol{X} / \partial t$ is parallel to the string, we must have
$$\boldsymbol{v}_\perp = \frac{\partial \boldsymbol{X}}{\partial t}$$
- In this parameterisation, we have
$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\frac{\partial X^\mu}{\partial \tau} \qquad \text{and} \qquad \mathcal{P}^{\sigma\mu} = -T_0 \frac{\partial X^\mu}{\partial s}\sqrt{1-\frac{v_\perp^2}{c^2}}$$
[The latter used to be the case at string endpoints].
- The string equation of motion was
$$\frac{\partial \mathcal{P}^{\tau\mu}}{\partial t} = -\frac{\partial \mathcal{P}^{\sigma\mu}}{\partial \sigma}$$
  - The $\mu = 0$ component gives
  $$\frac{\partial \mathcal{P}^{\tau 0}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{T_0}{c}\frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right) = 0$$
  For a small piece of string with $\mathrm{d}\sigma$, therefore, we find that the expression
  $$\frac{T_0}{c}\frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\mathrm{d}s$$
  Is constant. This represents the **energy** stored in that piece of string.
  - The space components give

$$\frac{\partial}{\partial\sigma}\left[T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\frac{\partial\boldsymbol{X}}{\partial s}\right]=\frac{\partial}{\partial t}\left[\frac{T_0}{c^2}\frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\boldsymbol{v}_\perp\right]$$

Using the time-component derived above, we can re-write this last step as

$$\frac{\partial}{\partial\sigma}\left[T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\frac{\partial\boldsymbol{X}}{\partial s}\right]=\frac{T_0}{c^2}\frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{\partial\boldsymbol{v}_\perp}{\partial t}$$

$$\frac{\partial}{\partial s}\left[T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\frac{\partial\boldsymbol{X}}{\partial s}\right]=\frac{T_0}{c^2}\frac{1}{\sqrt{1-\frac{v_\perp^2}{c^2}}}\frac{\partial\boldsymbol{v}_\perp}{\partial t}$$

Comparing to a non-relativstic string with $s \approx x$, we see that they match.

- Now is time to actually find a σ parameterisation. Consider the equation above, but slightly re-written

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{X}}{\partial t^2}=\frac{\sqrt{1-\frac{v_\perp^2}{c^2}}}{\frac{\mathrm{d}s}{\mathrm{d}\sigma}}\frac{\partial}{\partial\sigma}\left[\frac{\sqrt{1-\frac{v_\perp^2}{c^2}}}{\frac{\partial s}{\partial\sigma}}\frac{\partial\boldsymbol{X}}{\partial\sigma}\right]$$

And now, if we let

$$A(\sigma)=\frac{\frac{\mathrm{d}s}{\mathrm{d}\sigma}}{\sqrt{1-\frac{v_\perp^2}{c^2}}}=1$$

We get

$$\boxed{\frac{1}{c^2}\frac{\partial^2\boldsymbol{X}}{\partial t^2}=\frac{\partial^2\boldsymbol{X}}{\partial\sigma^2}}$$

Which is a wave equation, and we have

$$\mathrm{d}\sigma=\frac{\mathrm{d}s}{\sqrt{1-\frac{v_\perp^2}{c^2}}}=\frac{1}{T_0}\mathrm{d}E\Rightarrow\sigma_1=\frac{E}{T_0}$$

Which means that we've sub-divided the string into bits of energy. This condition is equivalent to

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2 + \frac{1}{c^2}v_\perp^2 = 1$$
$$\left(\frac{\partial \boldsymbol{X}}{\partial \sigma}\right)^2 + \frac{1}{c^2}\left(\frac{\partial \boldsymbol{X}}{\partial t}\right)^2 = 1$$

- And so we finally get

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{\partial X^\mu}{\partial \tau} \qquad \text{and} \qquad \mathcal{P}^{\sigma\mu} = -T_0\frac{\partial X^\mu}{\partial \sigma}$$

2. General Motion Open & Closed Strings

- Solving the wave equation, we get

$$\boldsymbol{X}(t,\sigma) = \frac{1}{2}\left[\boldsymbol{F}(ct+\sigma) + \boldsymbol{G}(ct-\sigma)\right]$$

  We can add the two parameterisation conditions to each other, and get

$$\boxed{\left(\frac{\partial \boldsymbol{X}}{\partial \sigma} \pm \frac{1}{c}\frac{\partial \boldsymbol{X}}{\partial t}\right)^2 = 1}$$

  Which gives $\left|\boldsymbol{F}'\right|^2 = 1$. This implies that **$\boldsymbol{u}$ is a distance parameter along the string**.
- For an open string, the relevant conditions are free conditions at the end of the string $\mathcal{P}^{\sigma\mu} = 0$.
- For a closed string
  - We take derivatives of $\boldsymbol{X}$ with respect to t and σ, form linear combinations and use the parameterisation conditions above.
  - Then, use the periodicity conditions and differentiate with respect to u and v to find periodicity conditions for $\boldsymbol{F}$ and $\boldsymbol{G}$.

  This means that $\boldsymbol{F}'$ and $\boldsymbol{G}'$ lie on the surface of the sphere (they have unit modulus and are periodic). This means that at some time, they'll cross, and
  - $\partial \boldsymbol{X} / \partial t = c\boldsymbol{F}'(u_0)$ – and this implies that the speed these is the speed of light, in the direction of $\boldsymbol{F}'$.
  - We then get $\boldsymbol{X}(t_0,\sigma) = \boldsymbol{X}_0 + \frac{1}{2}\left(\sigma - \sigma_0\right)^2 \boldsymbol{T} + \frac{1}{3!}\left(\sigma - \sigma_0\right)^3 \boldsymbol{R} + \cdots$, where $\boldsymbol{T} = \partial^2 \boldsymbol{X} / \partial \sigma^2 (t_0, \sigma_0)$ and $\boldsymbol{R} = \partial^3 \boldsymbol{X} / \partial \sigma^3 (t_0,\sigma_0)$.